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Optimal formula instruments

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Abstract

When estimating the effects of treatments defined by complex formulas, researchers often use simple functions of exogenous shocks as instruments. A leading example is "simulated instruments" for public policy eligibility, which capture variation in state-level policy generosity. We show how more powerful instruments can be constructed by incorporating heterogeneous shock exposure while using a recentering procedure to avoid bias. We characterize the asymptotically efficient instruments in this class and propose an algorithm for constructing feasible approximations to them. Compared to a simulated instrument approach, our approach yields a 44% smaller standard error on the private insurance crowd-out effect of Medicaid enrollment from the 2014 Affordable Care Act expansions.

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1 Introduction

Many economic variables are given by complex formulas, incorporating multiple sources of variation. Examples include an individual's eligibility for a public program like Medicaid or their level of unemployment insurance benefits, both of which are functions of state-level policy decisions as well as various individual characteristics (e.g. family structure, household income, or work history). When estimating the causal effects of such treatments, researchers often construct instrumental variables (IVs) as simple functions of only one source of variation. For example, the influential "simulated instrument" approach of Currie and Gruber (1996a, 1996b) leverages state-level policy shocks by constructing an index of Medicaid generosity which is then used to instrument an individual's Medicaid eligibility.[1] These instruments are valid when the policy shocks are exogenous and the chosen function of them predicts eligibility, at least somewhat.

This paper shows how more powerful instruments can be constructed and used in such settings. Intuitively, power gains can come from the instrument predicting the treatment better. This can be achieved by constructing the instrument as a treatment prediction that is a function (or "formula") of not only the exogenous shocks but also other observables capturing observations' differential shock exposure. Such treatment predictions need not be valid instruments, because of the nonrandom observables used in their construction. However, following the insight of Borusyak and Hull (2023), this problem can be addressed by "recentering" the treatment prediction: i.e., subtracting its expectation over the exogenous shocks, holding fixed the other observables. The class of valid formula instruments is therefore much broader than functions of exogenous shocks only.

We first characterize optimal formula instruments in a general setting. We show that the asymptotically efficient IV involves three steps: obtaining the best predictor of the treatment from both the shocks and other predetermined measures of shock exposure, recentering it to avoid bias, and adjusting for the error term's dependence on shock exposure and for heteroskedasticity. This result does not require *iid* data, covering a wide range of empirical settings where both observed and unobserved shocks—potentially varying at different "levels"—affect many observations jointly.

We then propose an algorithm to approximate optimal IVs in practice, focusing on the first two steps: obtaining the best treatment predictor and recentering it. While implementing both steps nonparametrically may be feasible in some settings, in general they represent a high-dimensional problem that may be impractical or infeasible—especially in non-*iid* data. Instead, we propose using knowledge the researcher has on the treatment formula as well as the "design" (i.e., data-generating process) of the exogenous shocks. First, the researcher predicts the treatment from the shocks and other observables which enter the treatment formula, setting any unobserved or endogenous components of the formula to a base value (such as zero). When there are no unobserved or endogenous components, this prediction is the treatment itself. Second, the researcher recenters this prediction by drawing counterfactual sets of exogenous shocks, following Borusyak and Hull

[1] See also Cullen and Gruber (2000) and East and Kuka (2015) for simulated instruments in the unemployment insurance setting. Other simulated instrument applications include Cohodes et al. (2016), Frean et al. (2017), Brown et al. (2018), and Hackmann (2019).

(2023). Optionally residualizing the recentered prediction on covariates yields an approximation to the optimal instrument, up to the heteroskedasticity adjustment that is not popular in practice.

Specializing this algorithm to the program eligibility setting yields a likely improvement over the conventional simulated instruments approach. If program eligibility is fully determined by predetermined observables and the policy shock, our proposed approach involves instrumenting an individual's eligibility with the difference between her actual eligibility (as the best possible predictor) and her expected eligibility, where the expectation is taken for each individual across the exogenous policy shocks that could have been realized (such as realized policies of comparable states). If program eligibility depends on other variables, such as income that can respond endogenously to policy shocks, the researcher can imperfectly predict eligibility using lagged income and recenter that prediction instead.

The proposed algorithm can also be helpful in many popular settings where researchers construct formula instruments as certain treatment predictions without formal theoretical justification. Consider, for instance, Boustan et al. (2013) who construct an instrument for the change in the Gini index of regional income distribution. They use exogenous national shocks to incomes of different population groups along with initial local shares of those groups. From these data, they measure and use as an IV the Herfindahl change that would follow if the exogenous income shocks were the only changes that took place, in a nonlinear version of a Bartik (1991) instrument. While this instrument is generally not valid without recentering, our results show that the recentered version of their IV is approximately optimal. Notably, conventional results on optimal instruments with *iid* data would not be applicable in their context, in which the same national shocks affect all regions simultaneously. Our results similarly justify the shift-share instrument construction proposed informally by Borusyak et al. (2025a) and recentered instruments for changes in market access due to transportation upgrades proposed by Borusyak and Hull (2023).

We then demonstrate the power gains empirically, in an application to the partial 2014 Medicaid expansion from the Affordable Care Act (ACA). A recentered IV which incorporates variation in individuals' exposure to state expansion decisions yields a 44% smaller standard error on the private insurance crowdout effect of Medicaid enrollment, compared to a more conventional simulated IV approach leveraging expansion shocks only. These power gains are robust to different IV specifications and assumptions on the expansion shock design. Monte Carlo simulations show the minimum detectable effects of recentered IV are roughly three times smaller than those of simulated IV.

Our theoretical results build on the classical literature on efficiency bounds and optimal instruments in linear and partially linear models. For linear models without functional nuisance parameters, Chamberlain (1987) characterizes the semi-parametric efficiency bound (SEB) and gives an IV estimator that achieves the bound. More closely connected to our setting is the partially linear model of, e.g., Robinson (1988): it is the special case when the outcomes, treatments, shocks, and other observed characteristics are *iid*, with the role of the nuisance function played by the expectation of the error term given the characteristics. There Chamberlain (1992) characterizes the SEB, Newey (1989) proposes an estimator that achieves this bound in the special case when the treatment is exogenous, and Ai and Chen (2003) derive an efficient sieve-based estimator in the general case. Against this backdrop, our theoretical contribution is to characterize optimal instruments in a broad class of non-*iid* settings including shift-share IV and many other popular designs (Borusyak et al. 2025b). We develop a proof technique for such settings: we characterize the estimator that minimizes an approximation to the estimator variance in finite samples and verify that the approximation is accurate in large samples under suitable regularity conditions.[2] We further show that our optimal IV attains the Chamberlain (1992) SEB in the *iid* case.

Most directly, we build on Borusyak and Hull (2023) who show how valid IVs can be derived, through recentering, from any formulas combining exogenous shocks and non-random measures of shock exposure. Here we show *which* formulas are theoretically best to recenter and use as IVs, and propose a general algorithm for approximating them in practice. Our application to Medicaid eligibility effects similarly builds on an example that Borusyak and Hull (2023) use to motivate instrument recentering as a means to avoid omitted variables bias; here, we use it to motivate and illustrate efficiency gains with our optimal IV approximation.

In contemporaneous work, Coussens and Spiess (2021) characterize optimal instruments given by interactions of a single observation-specific shock with predetermined characteristics, highlighting the benefit of interacting the shock with the complier status of the individual. Our result nests theirs, with individual's exposure to the shocks generalizing the complier status beyond binary instruments and allowing multiple shocks to affect the same observation's treatment and the same shocks to affect multiple observations' treatments. While Coussens and Spiess (2021) focus on learning the compliance status nonparametrically in *iid* data, we leverage *a priori* information on the structure of the treatment to approximate the optimal IV.

We organize the rest of the paper as follows. The next section builds intuition with a simple example in the simulated instrument setting. Section 3 develops the theoretical results while Section 4 demonstrates them in an application. Section 5 concludes. Additional theoretical results are given in Appendix A; Appendix B provides details for the application. Proofs are given in Appendix C.

2 Motivating Example

Consider estimating the causal effect of eligibility for a public program like Medicaid on an outcome like program takeup or later health. Formally, we consider a simple causal model of

$$y_i = \beta x_i + \varepsilon_i,$$

relating outcome y_i for individual i to her Medicaid eligibility x_i. Here ε_i denotes the potential outcome that individual i would see when ineligible for Medicaid (i.e., when $x_i = 0$) and β is the

[2] The analysis of efficiency with non-*iid* data dates back at least to generalized least squares estimation (Aitken 1935); optimal instruments under conditional moment restrictions in non-*iid* data have been most thoroughly studied with time-series data (e.g., Hansen (1985); see Anatolyev (2007) for a review). Our proof techniques bypasses the need for additional structure in such analyses, such as an assumption of Gaussianity in the examples in Section 6.4 of Newey (1990).

causal parameter of interest. We wish to estimate β while allowing for the possibility of endogenous eligibility: i.e., that x_i and ε_i are correlated. To focus on estimation efficiency, we assume here that the causal effect β is homogeneous.

Medicaid eligibility can be represented as a formula which incorporates state-level government policy and individual characteristics that determine an individual's exposure to different polices. To formalize this, let c_i be a vector of characteristics for individual i (e.g., family structure and income), let $s(i) \in \{1, \ldots, 50\}$ index i's state of residence, and let g_k be the Medicaid policy in state k formalized as the set of family types and income combinations that make one eligible for Medicaid in that state. Eligibility is then given by:

$$x_i = \mathbf{1}\left[c_i \in g_{s(i)}\right].$$

Consider estimation of β in an idealized scenario where Medicaid policies are drawn in a natural experiment: i.e., randomly from some pool of potential policies.[3] Formally we suppose the g_k are drawn from some distribution, independently of all c_i, $s(i)$, and ε_i. We do not assume the other determinants of eligibility are exogenous (i.e., unrelated to ε_i): individuals with certain characteristics or living in particular states may have systematically higher or lower potential outcomes.[4] Hence, despite the exogeneity of state policies, ordinary least squares (OLS) estimation of β is likely biased. We need an instrument for eligibility, which is uncorrelated with ε_i but correlated with x_i.

Simulated instruments leverage the exogenous policy shocks by constructing an IV as a function of state policy only: $z_i = f\left(g_{s(i)}\right)$. The function $f(\cdot)$ is chosen to make the instrument powerful—specifically, to make z_i a strong predictor of the eligibility treatment. The strongest predictor that only varies through state policy is $\mathbb{E}\left[x_i \mid g_{s(i)}\right]$, which can be interpreted as the average generosity of i's state policy. Currie and Gruber (1996a, 1996b) propose a simple approximation to this predictor. They build a large and nationally-representative group of individuals $j = 1, \ldots, J$, simulate individuals' eligibility under each state policy $\bar{g}$, and define $f(\bar{g})$ as the fraction of individuals who would be eligible under that policy:

$$f(\bar{g}) = \frac{1}{J}\sum_{j=1}^{J} \mathbf{1}\left[c_j \in \bar{g}\right].$$

The simulated instrument $z_i = f\left(g_{s(i)}\right)$ is a fixed function of the exogenous policy in an individual's state and is therefore uncorrelated with ε_i.[5] It is nevertheless correlated with x_i because we expect the eligibility of any given individual to be higher in states where the policy is more generous.

A drawback of such instruments, which limits their power, is that they discard all within-state

[3] This does not presume that any policy could arise: for instance, the pool may only include potential policies that prioritize the poor.

[4] This captures a central concern in the simulated instruments literature. For example, Gruber (2003, p.47) writes that "eligibility is a function of a variety of factors about individuals that might also be correlated with their underlying behavior, such as income, family structure, or age."

[5] Formally, for any fixed $f(\cdot)$, $\mathbb{E}\left[z_i \mid s(i), \varepsilon_i\right] = \mathbb{E}\left[f\left(g_{s(i)}\right) \mid s(i), \varepsilon_i\right] = \int f(\bar{g})dG(\bar{g}) = \mathbb{E}\left[z_i\right] \equiv \mu_z$ where $G(\cdot)$ is the distribution of potential Medicaid policies. Thus $\operatorname{Cov}\left[z_i, \varepsilon_i\right] = \mathbb{E}\left[\left(z_i - \mu_z\right)\varepsilon_i\right] = \mathbb{E}\left[\left(\mathbb{E}\left[z_i \mid s(i), \varepsilon_i\right] - \mu_z\right)\varepsilon_i\right] = 0$.

variation in eligibility due to c_i. This may seem unavoidable, as such variation is non-random and using it may introduce bias. For example, one might consider constructing an instrument to approximate $\mathbb{E}\left[x_i \mid g_{s(i)}, c_i\right]$ instead of $\mathbb{E}\left[x_i \mid g_{s(i)}\right]$; the former has a stronger first-stage correlation with x_i, so using this IV would likely produce a smaller standard error than z_i. However, this instrument may be correlated with ε_i through some characteristics in c_i, making the IV estimates biased. Indeed, here $\mathbb{E}\left[x_i \mid g_{s(i)}, c_i\right] = x_i$, since eligibility is fully determined by $g_{s(i)}$ and c_i. Using it as an instrument is thus equivalent to OLS estimation, with the same bias concerns as before.

The main practical insight of this paper is that improved predictions of formula treatments, including x_i itself, can be used to construct valid and more powerful instruments. Consider:

$$\tilde{z}_i = \underbrace{\mathbf{1}[c_i \in g_{s(i)}]}_{=x_i} - \underbrace{\frac{1}{50}\sum_{k=1}^{50}\mathbf{1}[c_i \in g_k]}_{\equiv \mu_i}.$$

The first term of $\tilde{z}_i$ is individual i's eligibility, determined by both her characteristics c_i and the realized policy draw in her state $g_{s(i)}$. The second term is her *expected* eligibility over policy draws. This μ_i is derived from a thought experiment in which the realized policies are randomly reshuffled across the 50 states, since each permutation of $(g_1, \ldots, g_{50})$ is as likely to have been realized when policies are drawn randomly from some set. On average, an individual's eligibility across such permutations equals the share of states in which her characteristics would make her eligible. Borusyak and Hull (2023) show that the *recentering* of x_i by μ_i makes $\tilde{z}_i$ a valid instrument. Intuitively, IV regressions that use $\tilde{z}_i$ compare individuals who have more Medicaid eligibility than expected given the policy experiment (i.e., those with $\tilde{z}_i > 0$) to those with less-than-expected eligibility (with $\tilde{z}_i < 0$). Since this delineation is by chance, driven only by the random policy shocks, such IV regressions are free from bias.[6] In the next section we discuss different strategies for recentering that follow from different assumptions of how the shocks are drawn.

The recentered instrument $\tilde{z}_i$ is likely more powerful than the simulated instrument z_i, as it is more predictive of the treatment. By construction, the policy generosity measure $f\left(g_{s(i)}\right)$ is not tailored to an individual's exposure to the policies, limiting z_i's correlation with x_i. In contrast, $\tilde{z}_i$ is perfectly correlated with x_i conditional on c_i (i.e., among individuals with the same family structure and income but who reside in different states). Only individuals for whom the policy variation is relevant are in the effective sample with $\tilde{z}_i$, since $\tilde{z}_i = 0$ for any individual who is, e.g., so rich that they are not eligible under any policy or so poor that they are eligible under all policies. The next section formalizes the sense in which such instruments likely yield precise estimates of β, and proposes a general algorithm for producing them from formula treatments like x_i.[7]

Power when using $\tilde{z}_i$ may be further increased by including functions of c_i (e.g. family size or some income bins) as controls. Such controls soak up residual variation in ε_i while not affecting the

[6] More formally, Borusyak and Hull (2023) show μ_i and x_i have the same covariance with ε_i so $\mathbb{E}\left[(x_i - \mu_i)\,\varepsilon_i\right] = 0$.

[7] Precision gains may also yield additional benefits: the more powerful recentered instrument may be less prone to finite-sample weak-IV bias and sizable asymptotic bias from small violations of instrument exogeneity (Wooldridge 2002, Ch. 5.2.6), and may also yield more powerful placebo tests (as we show in the application).

instrument's first stage (because $\tilde{z}_i$ is uncorrelated with all predetermined characteristics as well as with ε_i), typically increasing estimation efficiency.[8] The next section formalizes this logic by showing how the theoretically most efficient IV estimation of β involves such residual adjustment in addition to forming the recentered best predictor $\tilde{z}_i$; we add an optional step to the algorithm that involves such adjustment. Although extra covariates can increase the power of simulated IV estimation too, the two approaches do not coincide even when controlling for c_i flexibly.[9]

Before proceeding, we note that while x_i in this example is fully determined by the exogenous policy shocks g_k and predetermined variables c_i and $s(i)$, similar recentered IVs may be constructed for treatments with endogenous components. For example, suppose an individual's income u_i is relevant to Medicaid eligibility, $c_i = (u_i, \tilde{c}_i)$, but is not predetermined: income may respond to the policy shocks as individuals change employment. In this case $\tilde{z}_i$ will not be a valid IV, as recentering by μ_i will not account for the endogenous response of c_i to $g_{s(i)}$. Still, a strong predictor of x_i can be formed from its formula: one can compute the predicted eligibility based on the realized policy shocks, other characteristics $\tilde{c}_i$, and an earlier measure of income u_{0i} that replaces u_i. This approximation of $\mathbb{E}\left[x_i \mid g_{s(i)}, u_{0i}, \tilde{c}_i\right]$ can then be recentered, as before, to obtain a valid and likely powerful instrument. The same logic applies when an individual's state of residence $s(i)$ is not predetermined, such as when individuals move in response to the policy shocks. Cases where some characteristics relevant for eligibility are unobserved can also be handled similarly.[10]

3 Theory

We now develop general theory for optimal formula instruments. Section 3.1 introduces the setting and the class of valid recentered instruments. Section 3.2 derives the recentered instrument that is asymptotically most efficient, while Section 3.3 develops an algorithm for obtaining feasible approximations to the optimal IV.

3.1 Setting

An outcome y_i and treatment $x_i = h_i(g, w, u)$ are observed for a set of units $i = 1, \dots, N$. Here $h_1(\cdot), \dots, h_N(\cdot)$ is a set of known functions, $g = (g_1, \dots, g_K)$ is a set of observed shocks (potentially varying at a different level than i), w is a set of observed predetermined variables, and u is another set of variables (potentially unobserved and also potentially varying at different levels).[11]

[8] If the included controls linearly span expected eligibility μ_i, using $\tilde{z}_i$ as the instrument is numerically equivalent to using OLS estimation on x_i with the same controls—slightly simplifying implementation (Borusyak and Hull 2023).

[9] The two approaches would coincide if policy generosity was measured for each combination of characteristics in c_i separately, which amounts to interacting flexible functions of c_i with eligibility in addition to controlling for them. This strategy, however, is only feasible if c_i is discrete with a sufficiently small number of distinct values, which is not the case in most applications of simulated IV (see., e.g., Gruber (2003, p.47)).

[10] For example, we might be interested in the effects of Medicaid *enrollment* rather than eligibility and not observe an individual's compliance status (i.e. whether they would take up Medicaid when eligible). This u_i can be ignored when predicting x_i from the shocks and predetermined observables, e.g. by presuming that all individuals are compliers. If compliance status can be partially predicted, incorporating this may yield further precision gains.

[11] In some settings, like the motivating Medicaid example, it may be more natural to write $x_i = h(g, w_i, u_i)$ for an "anonymous" formula $h(\cdot)$ and observation-specific inputs w_i and u_i. Not all settings with formula treatments are

This formulation of x_i is so far without loss of generality; Assumption 1, below, makes it restrictive by introducing substantive distinctions between g, w, and u (in particular g will be assumed exogenous, as in the motivating example). Since multiple observations may be exposed to the same observed and potentially unobserved shocks, we do not make any assumptions of independently or identically distributed (*iid*) data and instead work with a fixed sample of size N; that is, we consider $(x_1, \ldots, x_N, y_1, \ldots, y_N, g, w, u)$ as drawn from some joint distribution.[12] The results apply to observations randomly sampled from some population as well. Some later assumptions on the existence of consistent and asymptotically regular recentered IV estimators will require the number of shocks K to grow with N.

A causal effect or structural parameter β relates the outcome to treatment by

$$y_i = \beta x_i + \varepsilon_i, \tag{1}$$

where ε_i is an unobserved error. Here we assume the outcome model is linear with a constant effect.[13] For notational simplicity, we do not include a constant in the model (as when, e.g., $\mathbb{E}\left[\sum_i \varepsilon_i\right] = 0$); Appendix A.1 generalizes our results to settings where y_i and x_i are demeaned or otherwise residualized on a vector of predetermined covariates.[14]

To estimate β, we look for an instrument $z = (z_1, \ldots, z_N)'$ that satisfies an exogeneity condition:

$$\mathbb{E}\left[\frac{1}{N}\sum_i z_i \varepsilon_i\right] = 0 \tag{2}$$

as well as a relevance condition $\mathbb{E}\left[\frac{1}{N}\sum_i z_i x_i\right] \neq 0$, such that $\beta = \mathbb{E}\left[\sum_i z_i y_i\right] / \mathbb{E}\left[\sum_i z_i x_i\right]$.[15] The IV estimator corresponding to z is then given by $\hat{\beta}[z] = \left(\sum_i z_i y_i\right) / \left(\sum_i z_i x_i\right)$.

While the class of IV estimators is restrictive relative to the more general class considered in, for instance, Chamberlain (1992), our non-*iid* analysis makes it much less restrictive than it may seem. In particular, by the Frisch-Waugh-Lovell theorem, it includes IV estimators with covariates since z_i can be chosen to be an in-sample residualization of some instrument on those covariates. Indeed, as we show below, the optimal instrument involves such adjustment.

We form instruments by assuming the shocks are exogenous and relevant to the treatment. Specifically, we suppose that, conditional on the observed variables w, the outcome errors are mean-independent of g and the treatment is affected by g (or is otherwise dependent on it):

naturally written this way, however; see, e.g., footnote 12 of Borusyak et al. (2025b).

[12] We allow this distribution to be partly degenerate, such as when potential outcomes and predetermined observables are treated as fixed as in a more conventional design-based analysis (e.g., Athey and Imbens (2022)). See footnote 14 in Borusyak and Hull (2023) for more discussion.

[13] The constant effects assumption follows the optimal IV literature, facilitating an analysis of relative efficiency across different IV estimators—since different instrument constructions will generally identify different weighted averages of heterogeneous effects. Online Appendix D.1 characterizes these averages, showing they involve convex weights under natural generalizations of the classic independence, exclusion, and monotonicity assumptions in Imbens and Angrist (1994). We note that a weaker independence condition suffices under constant effects, which we impose below.

[14] See also Online Appendix D.2, which discusses an extension of our framework with y_i and x_i residualized on a set of fixed effects in panel data.

[15] We assume throughout that relevant expectations and other moments are well defined.

Assumption 1. *(Shock exogeneity):* $\mathbb{E}[\varepsilon_i \mid g, w] = \mathbb{E}[\varepsilon_i \mid w]$ *a.s. for all* i.

Assumption 2. *(Relevance):* $\mathbb{E}[x_i \mid g, w] \neq \mathbb{E}[x_i \mid w]$ *with positive probability for some* i.

In the basic Medicaid example, g is the vector of state eligibility policies and w contains the relevant characteristics c_i of all individuals along with their states of residence, $s(i)$. Then $h_i(\cdot)$ is the known algorithm which checks the eligibility of individual i using these inputs, and u is empty. If income changes in response to the policy, we include it in u while adding pre-period income to w.

Two remarks about Assumption 1 are due here. First, in some applications, the shocks are randomized after the realization of w; this makes them fully independent from w as well as from the errors under an exclusion restriction (that shocks only affect outcomes through the treatment), satisfying Assumption 1. Under full independence, the class of valid moment conditions is wider (Poirier 2017); by making the weaker and more conventional Assumption 1, we limit ourselves to IV estimators which are most popular in practice. Second, Assumption 1 allows ε_i to be arbitrarily correlated with the variables in w such that OLS estimation is generally biased even if u is empty.[16]

The class of instruments satisfying exogeneity under Assumption 1 can be sharply characterized. We refer to instruments constructed as $z_i = f_i(g, w)$ for a set of non-stochastic functions $\{f_i(\cdot)\}_{i=1}^N$ as *formula instruments*. We further call them recentered formula instruments, or just *recentered instruments*, if they are mean-zero given w:

$$\mathbb{E}[f_i(g, w) \mid w] = 0 \text{ a.s. for all } i. \tag{3}$$

Let $\mathfrak{R}$ denote the class of recentered instruments. This class is wider than it may appear: in particular, it includes estimators that use any formula instrument $p_i(g, w)$ while controlling for a vector $m_i(w)$ that linearly spans the "expected instrument" $\mathbb{E}[p_i(g, w) \mid w]$.[17] For example, the simulated IV estimator from Section 2 that has a constant expected instrument (see footnote 5) belongs to $\mathfrak{R}$ as long as the intercept is controlled for.

We then have the following result, letting ε be the vector collecting the ε_i:

Proposition 1. *Under Assumption 1, all recentered instruments satisfy the exogeneity condition (2). Moreover, only recentered instruments satisfy (2): for any distribution of* (z, g, w) *there exists a conditional distribution of* $\varepsilon \mid (z, g, w)$ *such that Assumption 1 holds but (2) fails, unless* z *is a deterministic function of* (g, w) *satisfying (3).*

[16] As in Borusyak and Hull (2023, footnote 14), we allow (ε, w) to be stochastic or (as is often assumed in design-based analyses of finite populations) fixed. With fixed (ε, w), Assumption 1 holds trivially but its role is shifted to the assumption, below, that the distribution of $g \mid w$ is known.

[17] To see this, first note that $\mathfrak{R}$ clearly includes instruments of the form $z_i = p_i(g, w) - \mathbb{E}[p_i(g, w) \mid w]$ with no controls. Moreover, estimators that use any such $z \in \mathfrak{R}$ as an IV while controlling for some set of functions $m(w) = (m_i(w))_{i=1}^N$ are also recentered IV estimators. This follows because, by the Frisch–Waugh–Lovell theorem, they can be represented as IV estimators that use the in-sample projection of z on $m(w)$ as the instrument. It is easy to see that $\mathfrak{R}$ is closed with respect to such residualizations. Furthermore, estimators that instrument with $p_i(g, w)$ while controlling for a $m(w)$ that linearly spans $\mathbb{E}[p_i(g, w) \mid w]$ also belong to $\mathfrak{R}$ since they are numerically equivalent to using $p_i(g, w) - \mathbb{E}[p_i(g, w) \mid w]$ as an IV while controlling for $m(w)$. Note that residualization has an additional impact on the error term, which the extended results in Appendix A.1 allow for.

The first part of the proposition follows Borusyak and Hull (2023) to show that recentered instruments are valid (here, under a weaker mean-independence condition). The second part is new, and highlights two ideas. First, formula instruments that are not recentered include some variation from w and are thus prone to exogeneity failures without further restrictions on the error term. Second, naturally, Assumption 1 would not justify the validity of instruments constructed from any other data besides the shocks and predetermined variables in w. With this characterization, we next look for the most efficient recentered instrument.

3.2 Optimal IV

We take a non-standard approach to deriving the asymptotically efficient instrument, given the non-*iid* setup. In this setting, there may be multiple asymptotic regimes under which recentered IV estimators are consistent, possibly with different convergence rates.[18] We therefore develop an approach that does not rely on a specific asymptotic approximation, in two steps. We first introduce a finite-population approximation to the variance of any recentered IV estimator, which we show is accurate in large samples whenever *some* asymptotic distribution exists.[19] We then find the recentered instrument that minimizes this approximation in finite samples.

The approximate variance of the IV estimator using recentered instrument $z \in \mathfrak{R}$ is defined as:

$$\mathcal{V}\left[z\right] = \frac{\mathrm{Var}\left[z'\varepsilon\right]}{\mathbb{E}\left[z'x\right]^2},$$

for $x = (x_1, \dots, x_N)'$, assuming the relevant variance and expectation exist. This expression represents the variance of $\frac{1}{N}z'\varepsilon / \mathbb{E}\left[\frac{1}{N}z'x\right]$; if the first-stage covariance $\frac{1}{N}z'x$ converges to a non-zero constant Π as $N \to \infty$, we expect $\mathcal{V}\left[z\right]^{-1}$ to be a good measure of estimation precision when N is large and appropriate regularity conditions hold.

To formalize this asymptotic approximation, we define a class of regular estimators of β.

Definition 1. *For non-random $r_N \to \infty$, a recentered IV estimator $\hat{\beta}\left[z\right]$ for $z \in \mathfrak{R}$ is regular if:*

(a) $r_N \left(\hat{\beta}\left[z\right] - \beta\right)$ *converges to a distribution with zero mean and variance* $0 < V < \infty$ *as* $N \to \infty$*;*

(b) $\frac{1}{N}z'x \xrightarrow{p} \Pi$ *for some* $\Pi \neq 0$*;*

(c) $\frac{1}{N}z'x$ *and* $\left(r_N \frac{1}{N} z'\varepsilon\right)^2$ *are uniformly integrable.*

We say such $\hat{\beta}\left[z\right]$ converges to β at rate r_N and refer to V as the asymptotic variance of $\hat{\beta}\left[z\right]$.

[18] For instance, recentered shift-share IV estimators can be consistent with fixed K under additional restrictions on the error term (Goldsmith-Pinkham et al. 2020) as well as in the $K \to \infty$ regime without these restrictions (Adão et al. 2019; Borusyak et al. 2022).

[19] Note that the just-identified IV estimator typically has no moments in finite samples (Wooldridge 2002, p. 101), justifying our focus on approximate and asymptotic variances. The asymptotic variance concept is most useful when the limiting distribution of $\hat{\beta}\left[z\right]$ is normal. However, it can be considered more broadly; in particular, a researcher with a quadratic loss function will generally value reductions in asymptotic variance outside the normal case.

The existence of regular recentered IV estimators can be established in different ways. Adão et al. (2019) give sufficient conditions for the asymptotic normality of estimators (i.e., condition *(a)*) for instruments that are linear in the shocks (i.e., shift-share instruments, discussed more below). In Appendix A.4 we give an alternative path to asymptotic normality with general recentered instruments that depend only on a small set of mutually independent shocks, with non-overlapping sets of shocks for most observation pairs. Both sets of conditions require the number of shocks to be large and for shock exposure to be dispersed, in that most observations are strongly exposed to only a small number of shocks (differentially across observations). Note, however, that a growing number of shocks is not generally necessary for the below results: as shown by Goldsmith-Pinkham et al. (2020) in the shift-share case, regular estimators may also exist when the errors are sufficiently weakly correlated and shock exposure is exogenous, even with few shocks.[20]

We then have the following result:

Proposition 2. *For any regular recentered IV estimator $\hat{\beta}[z]$ that converges to β at rate r_N with asymptotic variance V, $\mathcal{V}[z]$ provides a good large-sample approximation to the variance:*

$$\lim_{N\to\infty} \frac{\mathcal{V}[z]}{V/r_N^2} = 1.$$

This result justifies looking for the recentered IV estimator with the smallest approximate variance $\mathcal{V}[z]$. The following theorem characterizes the solution:

Theorem 1. *Suppose Assumption 1 holds, $\mathbb{E}[\varepsilon\varepsilon' \mid g, w] = \mathbb{E}[\varepsilon\varepsilon' \mid w]$ a.s., and this matrix is a.s. invertible. Consider the recentered instrument*

$$z^* = \mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]^{-1} \tilde{z} \quad \text{for } \tilde{z} = \mathbb{E}[x \mid g, w] - \mathbb{E}[x \mid w]. \tag{4}$$

The associated $\hat{\beta}[z^*]$ has the smallest approximate variance of all recentered IV estimators:

$$z^* \in \arg\min_{z\in\mathfrak{R}} \mathcal{V}[z],$$

with

$$\mathcal{V}[z^*] = \mathbb{E}\left[(\mathbb{E}[x \mid g, w] - \mathbb{E}[x \mid w])' \,\mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]^{-1} (\mathbb{E}[x \mid g, w] - \mathbb{E}[x \mid w])\right]^{-1}.$$

Three comments are due here. First, note that we impose weak conditions on the data-generating process: only that the shocks are exogenous in the sense of Assumption 1 and that the errors are not perfectly collinear.[21] While Theorem 1 also requires the second moments of ε to be independent of g given w (which holds when $g \perp\!\!\!\perp (\varepsilon, w)$, as when shocks are fully randomized), Appendix Theorem

[20] See also Propositions 3–5 of Borusyak and Hull (2023), which show a growing number of shocks is essentially necessary for consistency of recentered IV estimators absent restrictions on the correlation structure of the errors.

[21] The invertibility requirement precludes the existence of a function $c(w)$ satisfying $c(w)'\varepsilon = 0$ (almost-surely or at least conditional on a set of w of non-zero measure), which rules out two distinct cases. The first case is when the data have been residualized on the covariates $c(w)$, so that $c(w)'x = 0$ whenever $c(w)'\varepsilon = 0$; Appendix A.1 generalizes all results to that case. The second case is if $c(w)'x \neq 0$; then β can be revealed exactly for some realizations of w—an unrealistic scenario.

A1 provides a more cluttered expression for the optimal IV without that assumption. Second, we note that the approximate variance is minimized among all recentered IV estimators, regardless of whether they are regular, although $\mathcal{V}[z]$ need not be a useful object otherwise. Third, note that by Assumption 2 the instruments $\tilde{z}$ and z^* are non-trivial and $\mathcal{V}[z^*]$ is finite.

Theorem 1 builds on classic results on efficient estimation in *iid* data. In particular, Chamberlain (1987) characterizes optimal instruments in the *iid* linear model (see also Newey and McFadden (1994) for a different characterization that our proof leverages). Newey (1989, Section 5) derives an optimal estimator for the partially linear model with conditionally exogenous treatment, in which the functional nuisance parameter corresponds to $\mathbb{E}[\varepsilon \mid w]$ in our notation.[22] Ai and Chen (2003) propose a sieve-based efficient estimator in the general *iid* case. While the key advantage of Theorem 1 is that it applies in non-*iid* data, it is also instructive to specialize it to the *iid* case in which the semi-parametric efficiency bound (SEB) of Chamberlain (1992) applies. Appendix Proposition A4 shows that z^* attains this SEB asymptotically, suggesting that our limiting of the estimator class to recentered IV does not carry an asymptotic efficiency cost.[23]

Equation (4) reveals the structure of the optimal instrument z^*. It is based on the best predictor of treatment given the exogenous shocks and predetermined variables, $\mathbb{E}[x \mid g, w]$, recentered by its expectation over the shocks $\mathbb{E}[x \mid w] = \mathbb{E}[\mathbb{E}[x \mid g, w] \mid w]$. This recentered best predictor $\tilde{z}$ is then adjusted by $\mathbb{E}[\varepsilon\varepsilon' \mid w]^{-1}$. The following proposition unpacks this last step:

Proposition 3. *Let $\psi = \mathbb{E}[\varepsilon \mid w]$ and $\Omega = \mathrm{Var}[\varepsilon \mid w]$ be the conditional mean and variance of the errors, with Ω a.s. invertible. Then z^* from Theorem 1 can be written as*

$$z^* = \Omega^{-1}\left(\tilde{z} - \nu\rho\psi\right), \tag{5}$$

where $\rho\psi = \frac{\psi'\Omega^{-1}\tilde{z}}{\psi'\Omega^{-1}\psi}\psi$ is the Ω^{-1}-weighted projection of $\tilde{z}$ on ψ, and $\nu = \frac{\psi'\Omega^{-1}\psi}{1+\psi'\Omega^{-1}\psi} \in [0, 1)$.

This result shows that z^* is obtained by two sequential adjustments to the recentered best predictor $\tilde{z}$. First, $\tilde{z}$ is partially residualized on ψ. In many cases, $\nu \to 1$ when $N \to \infty$; by the Frisch-Waugh-Lovell theorem, the limit case of $\nu = 1$ amounts to controlling for ψ while instrumenting with $\tilde{z}$.[24] Second, there is an adjustment for the inverse conditional variance of the errors Ω^{-1}, as in conventional generalized least squares estimation.

Appendix Proposition A3 specializes Theorem 1 to the class of shift-share instruments that are linear in the shocks, for which feasible versions may be easier to obtain (as discussed below). The optimal instrument in this class resembles z^*, except with $\tilde{z}$ replaced by an optimal linear prediction of the treatment—i.e., the fitted value from a linear regression of x_i on the (recentered) vector of

[22] Conditional treatment exogeneity means $\mathbb{E}[\varepsilon - \mathbb{E}[\varepsilon \mid w] \mid x] = 0$, which here holds when $u = \emptyset$.

[23] Despite this, the estimators in Newey (1989) and Ai and Chen (2003) do not coincide with the one in Theorem 1 in *iid* data: both are weighted versions of the Robinson (1988) estimator, which involves non-parametric residualization of y and x on w, while $\hat{\beta}[z^*]$ involves partially controlling for $\mathbb{E}[\varepsilon \mid w]$ (see Proposition 3 and footnote 24 below).

[24] In general the residualization is partial for the same reason why, in conventional panel data settings, the efficient random effects estimator partially demeans the data by unit (e.g. Wooldridge (2002, p.286)). As with the unit-specific residual means in the random effects case, ψ is orthogonal to $\tilde{z}$ in expectation but not in the observed realization. Full residualization which imposes in-sample orthogonality is thus generally inefficient, as with the fixed effect estimator.

shocks, with coefficients that can vary across i and depend on w. This linear regression parallels the non-parametric regression of x_i on the shocks that yields $\mathbb{E}[x_i \mid g, w]$ in Theorem 1.

3.3 Feasible Approximations

Drawing on the above results, we propose an approach (Algorithm 1) for obtaining feasible approximations to z^* in three steps: approximating the best predictor $\mathbb{E}[x \mid g, w]$, recentering it, and (optionally) adjusting for $\mathbb{E}[\varepsilon\varepsilon' \mid w]^{-1}$. The algorithm leverages the researcher's knowledge of the formula of the treatment and the "design" (i.e. data-generating process) of the exogenous shocks. Consequently, it can be used even in situations when the conditional means and variances underlying z^* cannot be consistently estimated because the data are high-dimensional or non-*iid*. We compare the Algorithm 1 estimator with an alternative plug-in estimator using nonparametric estimates of the terms in z^* at the end of this subsection.

The first step is to approximate the best predictor $\mathbb{E}[x_i \mid g, w]$ with some $p_i(g, w)$ using the treatment formula $x_i = h_i(g, w, u)$. This is trivial when u is empty, as in the initial Section 2 example, since then $\mathbb{E}[x_i \mid g, w] = x_i \equiv p_i(g, w)$. Otherwise, we propose forming a predictor by integrating $h_i(g, w, u)$ over some hypothetical distribution of $u \mid g, w$. In the simplest case, if there is a typical or default value $\bar{u}$ of u, it can be plugged in to form a prediction of $p_i(g, w) = h_i(g, w, \bar{u})$. This is the approach suggested at the end of Section 2, where lagged income u_{0i} substitutes for the potentially endogenous contemporaneous income u_i.[25] Importantly, per Proposition 1, a misspecified distribution for $u \mid g, w$ may lead to an imperfect approximation of $\mathbb{E}[x_i \mid g, w]$. This inaccuracy will generally reduce efficiency, but it will not introduce bias so long as the approximation is recentered.

The second step is to recenter the predictor $p_i(g, w)$ using knowledge of the shock assignment process: formally, the distribution of $g \mid w$. Here we summarize several approaches, following Borusyak and Hull (2023) and Borusyak et al. (2025b). Recentering is straightforward when the shock assignment process is known, such as when g is generated by a randomized control trial with some experimental protocol (conditional on w, or more typically independent from w). The researcher can draw a large number of counterfactual shock vectors $g^{(j)}$ from this protocol, recompute the predictor $p_i(g^{(j)}, w)$ for each $j = 1, \ldots, J$, and average them: $\frac{1}{J}\sum_{j=1}^{J} p_i(g^{(j)}, w)$. This approximation to $\mathbb{E}[p_i(g, w) \mid w]$ can then be subtracted from $p_i(g, w)$, or controlled for in estimation.[26] The same steps can be followed in observational data when the elements of g are assumed to be exchangeable conditional on w, as with the exchangeable policy shocks in Section 2. The counterfactual $g^{(j)}$ can then be generated as a permutation of g, bypassing the challenge of specifying the distribution of the

[25] Berry et al. (1999) take a similar approach in the context of estimating demand for differentiated products, setting unobserved product quality shifters to zero when constructing a (non-recentered) instrument. Conlon and Gortmaker (2020) propose a refinement to this approach, integrating over the empirical distribution of estimated quality shifters rather than using a single default value.

[26] While a large J reduces noise in the instrument $p(g, w) - \frac{1}{J}\sum_{j=1}^{J} p(g^{(j)}, w)$, approximating μ_i with any finite number of draws $J \geq 1$ is sufficient for identification. To see this, redefine the shocks as $\tilde{g} = \left(g, g^{(1)}, \ldots, g^{(J)}\right)$. The original Assumption 1 implies Assumption 1 with respect to $\tilde{g}$. Moreover, $p(g, w) - \frac{1}{J}\sum_{j=1}^{J} p(g^{(j)}, w)$ is in $\mathfrak{R}$ redefined with respect to $\tilde{g}$ because $\mathbb{E}\left[p(g, w) - \frac{1}{J}\sum_{j=1}^{J} p(g^{(j)}, w) \mid w\right] = 0$ when $g \mid w$ and $g^{(j)} \mid w$ have the same distribution.

Algorithm 1 Feasible Approximation to the Optimal IV

0. Represent the treatment as a formula $x_i = h_i(g, w, u)$ for known functions $h_i(\cdot)$, where the inputs are exogenous shocks g, predetermined variables w, and possibly other variables u that are either unobserved or may respond to the shocks.

1. Construct a treatment prediction $p_i(g, w)$. If the treatment is a function of g and w only, set $p_i(g, w) = x_i$. Otherwise, set $p_i(g, w) = h_i(g, w, \bar{u})$ by replacing u with some base value $\bar{u}$.

   - *Optional:* Replace $p_i(g, w)$ with its linear approximation in g around some base value $\bar{g}$.

2. Form the instrument by recentering the prediction: $\tilde{z}_i = p_i(g, w) - \mu_i$ for $\mu_i = \mathbb{E}[p_i(g, w) \mid w]$.

   - To obtain μ_i, draw some (preferably large) number J of counterfactual shock vectors $g^{(j)}$ from the data-generating process of g (e.g., permutations of g when the shocks are exchangeable). Set $\mu_i = \frac{1}{J}\sum_{j=1}^{J} p_i(g^{(j)}, w)$.
   - If $p_i(g, w)$ is linear in the shocks, μ_i can be computed analytically from a specification for $\mathbb{E}[g \mid w]$ without a full specification of the shock data-generating process.

3. *Optional:* Residualize $\tilde{z}$ on predetermined variables (i.e., functions of w) that may predict the error ε, and reweight it by an estimate of $\mathrm{Var}[\varepsilon \mid w]^{-1}$.

shocks.[27] More generally, $g^{(j)}$ could be drawn from conditional permutations of shocks within—but not across—groups with the same shock distribution; we illustrate this approach below.[28]

As Borusyak et al. (2025b) note, recentering requires weaker yet assumptions when $p_i(\cdot)$ is linear in the shocks—as in "shift-share" instrument constructions. In that case, only the expectation of the shocks matters for $\mathbb{E}[p_i(g, w) \mid w]$ and thus needs to be modeled. In cases where that approach to recentering is more tenable, Borusyak et al. (2025b, Section 3.3) propose approximating a nonlinear $p_i(g, w)$ with its linear approximation in g around some fixed $g = \bar{g}$. This approach yields coefficients approximating those in the infeasible regressions of Appendix Proposition A3, discussed above. As with other imperfect predictions of $\mathbb{E}[x_i \mid g, w]$, the cost of such simpler recentering is a likely power loss from the linear approximation.

An optional third step in Algorithm 1 adjusts the $\tilde{z}$ approximation by an estimate of $\mathbb{E}[\varepsilon\varepsilon' \mid w]^{-1}$. Per Proposition 3, this involves two adjustments: residualizing on (i.e., controlling for) functions of w that predict ε, and weighting by an estimate of $\mathrm{Var}[\varepsilon \mid w]^{-1}$ (akin to feasible generalized least squares). In non-*iid* or high-dimensional settings, both steps may be challenging. Moreover, weighting by a flexible but noisy estimate of $\mathrm{Var}[\varepsilon \mid w]^{-1}$ may introduce bias, as in conventional settings (e.g., Angrist and Pischke 2008, p. 93).

A practical option is therefore to disregard the third step of Algorithm 1 and just instrument with the recentered $p_i(g, w)$ as an approximation to $\tilde{z}$. This approach has two formal justifications.

[27] Formally, this requires including the permutation class of g in w.

[28] See Borusyak and Hull (2023) for a discussion of other approaches to recentering, such as using a theory-based approximation to $\mu_i = \mathbb{E}[p_i(g, w) \mid w]$ (as in Abdulkadiroglu et al. (2017)) or using an estimated μ_i based on first-step estimates of the shock assignment process.

First, it is straightforward to show that the unadjusted $\tilde{z}$ has the highest correlation with the treatment among all recentered IVs:

Lemma 1. $\tilde{z} \in \arg\max_{z \in \mathfrak{R}} \frac{\mathbb{E}[z'x]^2}{\mathbb{E}[z'z]\mathbb{E}[x'x]}$.

This result immediately implies that $\tilde{z}$ produces the highest first-stage R^2, and so approximations to it should have favorable power. Second, because of this property, $\tilde{z}$ minimizes the worst-case estimator variance absent sharp information on the conditional error distribution:

Lemma 2. *Let $\bar{\lambda} \geq 0$ be some constant and let $\mathcal{E}$ be the class of random vectors $e = (e_1, \ldots, e_N)'$ such that the maximum eigenvalue of $\mathbb{E}[ee' \mid g, w]$ is bounded by $\bar{\lambda}$ uniformly over (g, w). Suppose the only knowledge a researcher has about ε is $\varepsilon \in \mathcal{E}$. Then $\tilde{z}$ is approximately minimax, in that:*

$$\tilde{z} \in \arg\min_{z \in \mathfrak{R}} \max_{e \in \mathcal{E}} \frac{\mathrm{Var}[z'e]}{\mathbb{E}[z'x]^2}.$$

This result captures the intuition that maximizing the first-stage R^2 is optimal when one has weak priors on the distribution of ε, and worries about making further incorrect adjustments.

Rather than skipping Step 3, the researcher could control for an arbitrary low-dimensional function of w that may be expected to predict $\mathbb{E}[\varepsilon \mid w]$. This may improve efficiency, though this is not guaranteed (see Appendix C.9 of Borusyak and Hull (2021) for a counterexample). Alternatively, when nonparametric estimation of the nuisance functions in z^* is possible, one could use a plug-in instrument $\hat{z}^* = \widehat{\mathbb{E}}[\varepsilon\varepsilon' \mid w]^{-1} \left(\widehat{\mathbb{E}}[x \mid g, w] - \mathbb{E}\left[\widehat{\mathbb{E}}[x \mid g, w] \mid w\right]\right)$ instead of the Algorithm 1 approximation. Appendix A.5 formalizes this strategy by deriving regularity conditions under which $\hat{\beta}[\hat{z}^*]$ has the same asymptotic distribution as $\hat{\beta}[z^*]$ and thereby attains the minimal asymptotic variance. This result requires that $\widehat{\mathbb{E}}[x \mid g, w]$ is recentered by $\mathbb{E}\left[\widehat{\mathbb{E}}[x \mid g, w] \mid w\right]$ rather than by a generic nonparametric estimate of $\mathbb{E}[x \mid w]$, showing the importance of Step 2 whether or not nonparametric estimation is additionally pursued.[29] Moreover, the result requires that the nonparametric estimates $\widehat{\mathbb{E}}[x \mid g, w]$ and $\widehat{\mathbb{E}}[\varepsilon\varepsilon' \mid w]$ are consistent and obtained via cross-fitting, which is challenging if not impossible outside simple *iid* settings.[30] Algorithm 1 does not involve non-parametric estimation or cross-fitting, which highlights its practical value especially in complex settings.

[29] This makes our plug-in estimator different from other nonparametric estimators, such as those studied by Ai and Chen (2003) and Chernozhukov et al. (2018) in the *iid* setting, which estimate $\mathbb{E}[x \mid w]$ fully nonparametrically but require nontrivial smoothness conditions on the nuisance functions or rate conditions on their estimates. Our results avoid such conditions, similarly to the results on double machine learning with a known propensity score (Wager 2025, Corollary 3.3).

[30] For example, Borusyak et al. (2025b, Section 6.1) show how nonparametric estimation of $\mathbb{E}[x \mid g, w]$ can be impossible in a shift-share setting. If all observations are exposed to the same vector of aggregate shocks, of which only one realization is observed, it is not possible to consistently estimate $\mathbb{E}[x \mid g, w]$ except by leveraging information on the formula and shock design, as in Step 1 of Algorithm 1. Similarly, when all observations are exposed to the same shocks, standard cross-fitting methods are not applicable. Even in the context of our Section 4 application, where individuals in the sample are drawn randomly but the shocks are assigned to a relatively small number of clusters (states), nonparametric estimation of $\mathbb{E}[x \mid g, w]$ would be challenging.

4 Application

We illustrate our approach by estimating the private insurance crowdout effects of Medicaid eligibility, using the partial Affordable Care Act (ACA) expansion in 2014. Section 4.1 describes the setting and adapts our theoretical results to this setting while Section 4.2 presents the results.

4.1 Setting and Estimators

In January 2014, many US states expanded Medicaid eligibility under the ACA to cover non-elderly adults with incomes up to 138% of the federal poverty level (FPL). This expansion did not cover all states since a 2012 Supreme Court decision (NFIB v. Sebelius, 567 U.S. 519) let individual state governors opt out of the more generous ACA coverage level. In practice, expansion decisions partially followed partisan lines: among the 43 states with less generous Medicaid policies in 2013, only 8 out of 30 states with Republican governors expanded while 11 out of 13 states with Democratic governors did.[31] Non-expansion states mostly kept their 2013 eligibility rules in place, though some increased coverage slightly. Expansion states mostly adopted the ACA's 138% FPL threshold, though some extended coverage further.

We use state expansion decisions as policy shocks for estimating Medicaid eligibility effects. To formalize this approach, consider a repeated cross section of individuals i observed in years $t(i) \in \{2013, 2014\}$ with states of residence $s(i)$. We write i's Medicaid eligibility, $x_i \in \{0, 1\}$, as:

$$x_i = h^{t(i)}(c_i, e^{2013}_{s(i)}, g_{s(i)}, e^{\Delta}_{s(i)}) \tag{6}$$

where c_i collects relevant individual characteristics (income, work status, and parental status), e^{2013}_k is state k's Medicaid eligibility policy in 2013, $g_k \in \{0, 1\}$ indicates whether or not state k expanded coverage under the ACA in 2014, and e^{Δ}_k includes other 2014 changes to Medicaid coverage (i.e. non-ACA coverage increases or ACA expansions beyond the 138% FPL threshold). These inputs are sufficient to determine individual i's eligibility, as formalized by the $h^{2013}(\cdot)$ and $h^{2014}(\cdot)$ functions.[32]

We assume the expansion shocks are exogenous, conditional on the political party of states' governors, when estimating eligibility effects in a difference-in-differences setup. Formally, let q_k be an indicator for state k having a Republican governor in 2013. We relate individual outcomes to eligibility in the repeated cross section by:

$$y_i = \beta x_i + \alpha_{s(i)} + \tau_{q_{s(i)}, t(i)} + \varepsilon_i, \tag{7}$$

where α_k and $\tau_{q,t}$ are state and party-by-year fixed effects. Following Assumption 1, we formalize shock exogeneity as $\mathbb{E}[\varepsilon_i \mid g, w] = \mathbb{E}[\varepsilon_i \mid w]$ where g collects the 43 expansion dummies and w collects

[31] See Frean et al. (2017) for more background on the partial Medicaid expansions and related ACA policy changes.

[32] Here, as in the initial Section 2 motivating example, we assume income (and other characteristics) does not respond to expansion decisions. In our repeated cross-section, we do not observe pre-period income so we cannot apply the extension at the end of that section. We allow, however, income (and other characteristics) to be arbitrarily correlated with potential outcomes.

all c_i, $s(i)$, $t(i)$, e_k^{2013}, and q_k.[33] This assumption is broadly consistent with earlier difference-in-differences analyses that compare outcome trends of expansion and non-expansion states before and after 2014, either without adjustment for state-level confounders (e.g. Miller and Wherry (2017)) or with adjustment for state party or other observables (e.g. Averett et al. (2019)). It is nevertheless a strong assumption, requiring states with potentially very different socioeconomic characteristics affecting ε_i to not be systematically more or less likely to expand coverage conditional on the governor's party. We thus relax it in robustness checks below, allowing expansion decisions to depend on other observables like household income. Note that we impose no assumptions on other coverage changes e_k^{Δ}, which we collect in u.

Our baseline design assumption is that the shocks are drawn from the same (unknown) distribution among states with the same-party governor. This view of the 2014 expansion decisions, as arising from a natural experiment, conforms with some earlier analyses (e.g. Black et al. 2019) and allows us to construct counterfactual $g^{(j)}$ vectors by permuting the shocks conditional on state party. That is, all $g^{(j)}$ with expansions in some 8 Republican-governor states and some 11 Democratic-governor states are valid shock counterfactuals. We again check sensitivity to richer designs below, allowing expansion probabilities to vary by additional state observables.

Following Section 2, we consider two IV specifications using these assumptions. First, we construct a simulated instrument $z_i = f^{t(i)}\left(g_{s(i)}\right)$ which only leverages the shock variation. Since the policy only changes in 2014, we set z_i to zero for all individuals in 2013 as well as those in non-expansion states. For the expansion states in 2014, the instrument measures the change in the policy generosity. Specifically, it equals the difference in the fraction of the 2014 nationally representative sample who would be eligible for Medicaid between two situations: had all states adopted the ACA's 138% FLP threshold vs. had all of them kept their 2013 policies intact. Exogeneity of the expansion shocks and the design assumption make this z_i uncorrelated with ε_i controlling for state and party-by-year fixed effects—controls that span the expected instrument. Note that estimation with this z_i is equivalent to instrumenting by the interaction of $g_{s(i)}$ and a 2014 indicator, as in an instrumented difference-in-differences specification.[34]

Next, we construct an approximation to the recentered best predictor $\tilde{z}_i$ by applying the first two steps of Algorithm 1 to the formula for Medicaid eligibility. In the first step, we predict eligibility from the exogenous policy shocks g and predetermined variables w, ignoring the non-ACA eligibility changes in u. In our notation, this involves replacing e_k^{Δ} with $\emptyset$ in equation (6) which yields a predicted eligibility of $p_i(g, w) = h^{t(i)}(c_i, e_{s(i)}^{2013}, g_{s(i)}, \emptyset)$. Second, we recenter this prediction by subtracting its expectation μ_i with respect to the expansion shocks, conditional on individual characteristics, 2013 policies, and the party of every state governor. Consistent with our design assumption, we account for differences in expansion probabilities by state party when taking

[33] State fixed effects can be justified here by a state-level unobservable which drives the expansion decision, has a time-invariant effect on the outcome, and is independent of w. Idiosyncratic policy preferences of the state's governor is one example; see Online Appendix D.2 for details. Party-by-year fixed effects are included to span the expectation of the simulated instrument, as discussed below.

[34] Indeed, given state and year fixed effects, the specific values of the instrument in the four cells (2013 vs. 2014 and treated vs. untreated states) are immaterial as long as z_i increases more in 2014 in the treated states.

the expectation:

$$\mu_i \equiv \mathbb{E}\left[p_i(g,w) \mid w\right] = \pi(q_{s(i)}) \cdot h^{t(i)}\left(c_i, e^{2013}_{s(i)}, 1, \emptyset\right) + (1-\pi(q_{s(i)})) \cdot h^{t(i)}\left(c_i, e^{2013}_{s(i)}, 0, \emptyset\right),$$

where $\pi(0) = 11/13$ and $\pi(1) = 8/30$ correspond to the fractions of expansion states among those with Democratic and Republican governors, respectively.

The recentered instrument $p_i(g,w) - \mu_i$ equals zero for all individuals in 2013, as well as individuals in 2014 whose characteristics make them eligible (or ineligible) with or without an ACA expansion in their state. It only varies among individuals in 2014 whose eligibility is affected by expansion decisions. We thus restrict estimation to these "exposed" individuals in 2014 and, in keeping with the difference-in-differences structure, the corresponding group of individuals in 2013 (i.e., those whose characteristics and state of residence would make them exposed to the expansion shocks in 2014)—what we call the exposed sample. We control for state fixed effects, year fixed effects, and the interaction of the state party indicator $q_{s(i)}$ and year, as in the simulated IV specification. With these controls and the restriction to the exposed sample, recentered IV estimation is again equivalent to instrumenting by the interaction of $g_{s(i)}$ and a 2014 indicator.[35]

We apply both IV strategies using data from the 2013 and 2014 American Community Surveys, using representative 1% samples of non-disabled U.S. adults (ages 21-64) residing in the 43 states eligible for ACA expansion in 2014. Appendix B.1 details the sample construction.

Our primary outcomes are indicators for Medicaid enrollment and for private insurance coverage. Effects on the former outcome capture takeup of the expanded Medicaid coverage; effects on the latter outcome capture how Medicaid eligibility crowds out other forms of insurance—an important policy parameter in the literature (e.g. Frean et al. 2017; Leung and Mas 2018). We also look at effects on an indicator for employer-sponsored insurance coverage, which is the most common type of private insurance and is the focus of the classic literature on Medicaid crowdout (e.g. Cutler and Gruber (1996)). In our setting, Medicaid may also crowdout private insurance obtained directly on ACA state health exchanges (as documented by Frean et al. (2017)). This would have different economic implications, as such crowdout would not typically have employment effects.

4.2 Results

Our recentered IV is much more predictive of actual Medicaid eligibility than the simulated IV. First-stage estimates in Table 1 show that the simulated IV predicts eligibility with a coefficient of 0.85 and a standard error of 0.11 (column 1) while the recentered IV in the exposed sample has a higher coefficient of 0.97 and a smaller standard error of 0.02 (column 2); the latter coefficient is statistically indistinguishable from one, as should be the case with the recentered best predictor.[36]

[35] This is because the recentered instrument can be written in the exposed sample as $g_{s(i)} \times \mathbf{1}\left[t(i) = 2014\right] - \pi(0) \times \mathbf{1}\left[t(i) = 2014\right] - (\pi(1) - \pi(0)) \times q_{s(i)} \times \mathbf{1}\left[t(i) = 2014\right]$ with the latter two terms absorbed by the controls.

[36] Throughout we report state-clustered standard errors. To address finite-sample concerns with a relatively small number of states, we report confidence intervals by a wild score bootstrap as suggested by Kline and Santos (2012) and use them for hypothesis testing. This computationally efficient approach requires inverting bootstrapped test statistics, which generally makes confidence intervals asymmetric around the IV point estimate.

Table 1: Medicaid Application: First-Stage Effects on Eligibility

	(1)	(2)	(3)
Simulated IV	0.851		0.032
	(0.113)		(0.140)
	[0.585,1.108]		[-0.252,0.479]
Recentered IV		0.972	0.817
		(0.015)	(0.171)
		[0.941,1.014]	[0.394,1.161]
Partial R^2	0.012	0.799	0.103
Exposed Sample	N	Y	N
States	43	43	43
Individuals	2,397,313	421,042	2,397,313

Notes: This table reports first-stage coefficients for the two instruments described in the text: a conventional simulated instrument and a recentered prediction of Medicaid eligibility. Columns 1 and 3 estimate regressions in the full sample of individuals in 2013–14, while Column 2 restricts to the sample of individuals in both years whose characteristics and state of residence make them exposed to the partial ACA Medicaid expansion in 2014. All regressions control for state and year fixed effects and an indicator for Republican-governed states interacted with year. State-clustered standard errors are reported in parentheses; 95% confidence intervals, obtained by a wild score bootstrap, are reported in brackets. R^2 statistics partial out the controls.

The partial R^2 for the recentered IV is also dramatically higher (0.80, vs. 0.01 for the simulated IV). Column 3 of the table shows that just adding the recentered IV to the simulated IV specification increases the partial R^2 meaningfully (to 0.10) and renders the simulated IV coefficient small and statistically insignificant. All of these results are consistent with the recentered IV giving a better approximation to the recentered best predictor $\tilde{z}_i$.

This improved first-stage prediction translates to meaningful precision gains for the effects of eligibility on Medicaid and private insurance coverage. Columns 1 and 2 of Table 2, Panel A, show the standard error is 64% smaller with recentered IV vs. simulated IV (0.010 vs. 0.028) when estimating the effects of Medicaid eligibility on Medicaid enrollment. For crowdout effects, which take private insurance coverage as the outcome, standard errors are reduced by 70% (0.007 vs. 0.023; columns 3 and 4) from an insignificant simulated IV estimate to a significant recentered IV estimate. These estimates incorporate effects from both the conventional crowdout margin of employer-sponsored insurance as well as crowdout from ACA state health exchanges; in columns 5 and 6 we isolate crowd-out of employer-sponsored plans. Here neither simulated nor recentered IV yields significant estimates, though the latter is again much more precise.

In economic terms, the recentered IV estimates suggest a total private insurance crowdout rate of 32.1%, with a 7.2 percentage point increase in Medicaid coverage offset by a 2.3 percentage point decrease in private insurance coverage. This relative effect, reported in Panel B column 4 as the coefficient from an IV regression of private insurance coverage on Medicaid enrollment (instead of eligibility), is similar to the 42% crowd-out that Leung and Mas (2018) find in applying a difference-

Table 2: Medicaid Application: IV Estimates

	Has Medicaid		Has Private Insurance		Has Employer-Sponsored Insurance	
	Simulated IV (1)	Recentered IV (2)	Simulated IV (3)	Recentered IV (4)	Simulated IV (5)	Recentered IV (6)
Panel A. Medicaid Eligibility Effects						
Eligible	0.132	0.072	−0.048	−0.023	0.009	−0.009
	(0.028)	(0.010)	(0.023)	(0.007)	(0.014)	(0.005)
	[0.077,0.229]	[0.050,0.094]	[-0.104,0.001]	[-0.040,-0.008]	[-0.025,0.046]	[-0.020,0.003]
Panel B. Medicaid Enrollment Effects						
Has Medicaid			−0.361	−0.321	0.068	−0.125
			(0.165)	(0.092)	(0.111)	(0.061)
			[-0.740,-0.039]	[-0.549,-0.119]	[-0.137,0.332]	[-0.252,0.056]
Exposed Sample	N	Y	N	Y	N	Y
States	43	43	43	43	43	43
Individuals	2,397,313	421,042	2,397,313	421,042	2,397,313	421,042

Notes: Panel A of this table reports second-stage coefficients from the two IV regressions described in the text: one using a conventional simulated instrument and the other using as an instrument a recentered prediction of Medicaid eligibility. Columns 1, 3, and 5 estimate regressions in the full sample of individuals in 2013–2014, while Columns 2, 4, and 6 restrict to the sample of individuals whose characteristics and state of residence make them exposed to the partial ACA Medicaid expansion in 2014. All regressions control for state and year fixed effects and an indicator for Republican-governed states interacted with year. Panel B shows estimates from IV regressions which use an indicator for Medicaid enrollment as the endogenous variable, instead of an indicator for Medicaid eligibility. State-clustered standard errors are reported in parentheses; 95% confidence intervals, obtained by a wild score bootstrap, are reported in brackets.

in-differences specification to the 2014 Medicaid expansion.[37] However, we find minimal evidence for crowdout from employer-sponsored insurance plans even with our more powerful recentered IV. Instead, our estimates suggest crowdout arises from direct-purchase private insurance via state health exchanges (implying minimal employment effects). This aligns with the findings of Frean et al. (2017), who exploit multiple sources of ACA-induced policy variation with a simulated instrument (see also Courtemanche et al. (2017), Kaestner et al. (2017), and Maclean and Saloner (2019)). For both private insurance and employer-sponsored insurance, the standard errors of the recentered IV crowdout parameter estimates in Panel B are around 44% smaller than those of simulated IV.

Importantly, the recentered IV power gains are not a product of the relatively simple simulated IV specification. Online Appendix Table A1 shows that adding flexible controls for the individual characteristics which drive exposure to Medicaid expansions (specifically: full interactions of household income deciles, parental status, work status, and year) has little effect on the point estimates and standard errors for both estimators. Here the similar power of recentered IV with and without flexible controls suggests little added benefit from trying to approximate the adjustment for ψ in the optimal IV after approximating the recentered best predictor.

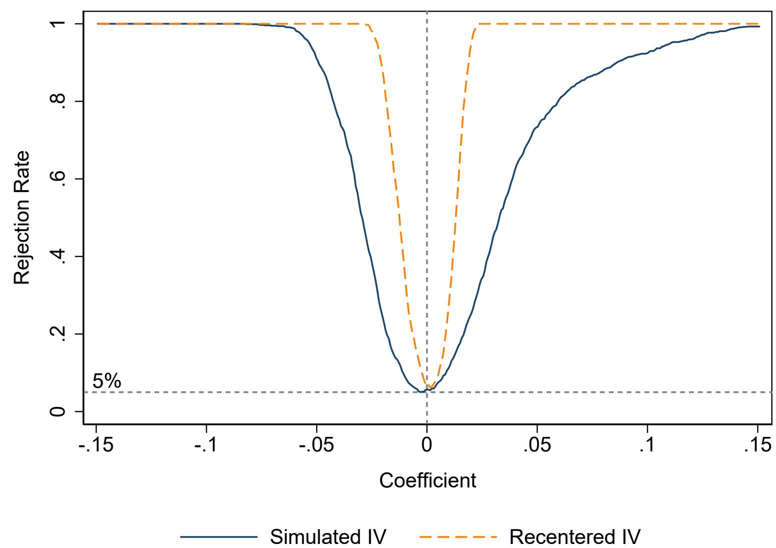
Three additional checks are presented in the Online Appendix. First, we check the assumption of expansion exogeneity with a pre-trend test that replaces the 2013-2014 difference-in-differences IVs with a comparable 2012-2013 analysis.[38] Table A2 shows that the increased precision of recentered IV lets us reject the null hypothesis of no pre-trends, highlighting how our approach can improve power of specification tests as well as effect estimates. While strictly speaking the rejection invalidates the identifying assumption underlying our approach (as well as those of earlier analyses), we proceed with it as the baseline since the magnitude of the placebo coefficient is reassuringly small (around 0.01–0.02) regardless of the outcome and the instrument. We then check sensitivity to allowing a state's decision to expand to depend not only on the governor's political party but also on the state's median household income and the 2012 rate of Medicaid coverage (specifically, by adding these three state characteristics as controls, interacted with year and state party dummies). Table A3 shows that estimated effects of eligibility remain very similar across these specifications. Third, we explore robustness to using the recentered IV without restricting to the exposed sample. Table A4 shows that this approach only yields power gains when the additional demographic controls (those from Table A1) or an indicator for being in the exposed sample interacted with year are included as covariates. We discuss the reason for this in Online Appendix D.4 by relating it to our general efficiency theory of Section 3.2.

[37] The corresponding simulated IV specification yields a private insurance crowd-out rate of 36.1%, reported in Panel B column 3, which is not statistically distinguishable from the recentered IV estimate ($p = 0.719$). Recentered and simulated IV also yield statistically indistinguishable estimates for the employer-sponsored insurance outcome ($p = 0.104$), reported in Panel B columns 5-6. In contrast, the recentered and simulated IV estimates which use Medicaid eligibility as the endogenous variable are statistically distinguishable, while they should have the same estimating according to the theory presented thus far. Online Appendix D.3 discusses how this pattern can be explained by measurement error in self-reported income and demographics entering the eligibility calculation. The specifications which use Medicaid enrollment as the endogenous variable are free from such bias.

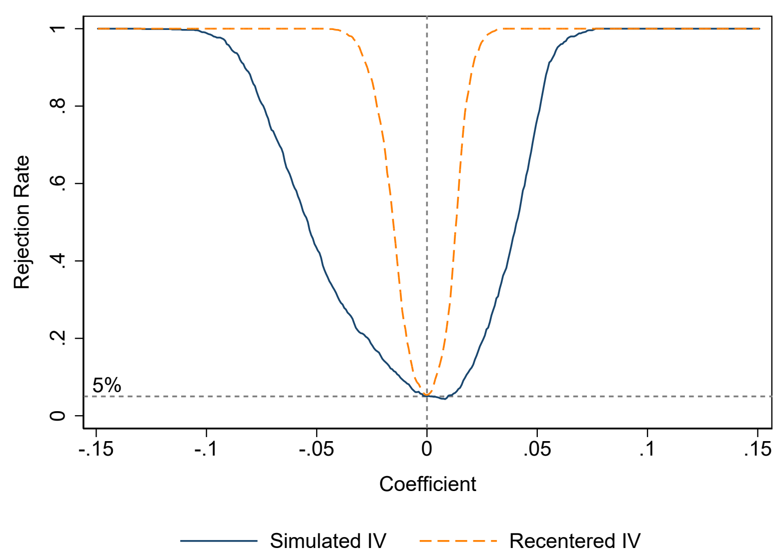
[38] Specifically, we replace 2013 individuals with 2012 individuals and replace 2014 individuals with 2013 individuals. We continue to construct the endogenous variable and instrument as an individual's Medicaid eligibility in 2013 and 2014 for comparability, and also keep all controls unchanged.

Figure 1: Medicaid Application: Simulated Power Curves

A. Has Medicaid



B. Has Employer-Sponsored Insurance



Notes: This figure plots the simulated rejection rates of the two IV regressions discussed in the text: one using a conventional simulated instrument and the other using as an instrument a recentered prediction of Medicaid eligibility. See Appendix B.2 for a description of the simulation procedure. Rejection rates are for nominal 5%-level tests of each coefficient based on wild score bootstraps, clustered by state. The true effect of zero is indicated by the dashed vertical line. The nominal 5% level of the tests is indicated by the dashed horizontal lines.

Large power gains from recentered IV are confirmed in a Monte Carlo study based on our baseline estimates. Figure 1 plots simulated power curves for the primary Medicaid enrollment outcome and the employer-sponsored insurance coverage outcome, while Online Appendix Figure A2 plots the simulated estimator distributions (Appendix B.2 details the simulation procedure). In this controlled environment the true causal effect and the shock assignment process are known, allowing us to verify that the recentered IV estimator is both close to unbiased and substantially more efficient than the simulated IV estimator. We find, for example, that the minimum detectable effects of recentered IV (i.e., the smallest null hypotheses which are rejected by a 0.05-size test with probability 0.8) are roughly three times smaller than those of simulated IV.

5 Conclusion

Many economic treatments are given by formulas which incorporate multiple sources of variation, only some of which are exogenous. Rather than discarding the other non-random variation, we show how it can be used to construct powerful formula instruments. By combining the known treatment structure with knowledge of the design of exogenous shocks, researchers can construct recentered instruments which strongly predict the treatment. We show how such recentered best predictors are formally justified, and how they can be further adjusted to approximate the asymptotically optimal formula instrument. Empirically, we show substantial power gains from using such instruments to estimate the crowdout effects of Medicaid enrollment—with standard errors 44% smaller than those from a conventional simulated instrument approach.

Importantly, while we have focused on the popular setting of simulated instruments, the insights of this paper may apply to a large class of formula treatments and instruments from a variety of fields. These include linear and nonlinear shift-share instruments, treatments capturing spillovers across social networks or geography, instruments based on centralized school assignment mechanisms, "free-space" instruments capturing access to mass media, and treatments or instruments leveraging weather shocks (Borusyak and Hull 2021). We expect this class to only increase as researchers find new and creative ways to exploit exogenous variation in complex treatments.

References

Abdulkadiroglu, Atila, Joshua D. Angrist, Yusuke Narita, and Parag A. Pathak. 2017. "Research Design Meets Market Design: Using Centralized Assignment for Impact Evaluation." *Econometrica* 85:1373–1432.

Adão, Rodrigo, Michal Kolesár, and Eduardo Morales. 2019. "Shift-Share Designs: Theory and Inference." *Quarterly Journal of Economics* 134:1949–2010.

Ai, Chunrong, and Xiaohong Chen. 2003. "Efficient estimation of models with conditional moment restrictions containing unknown functions." *Econometrica* 71:1795–1843.

Aitken, AC. 1935. "On least squares and linear combination of observations." *Proceedings of the Royal Society of Edinburgh* 55:42–48.

Anatolyev, Stanislav. 2007. "Optimal instruments in time series: A survey." *Journal of Economic Surveys* 21:143–173.

Angrist, Joshua D., and JS Pischke. 2008. *Mostly harmless econometrics: An empiricist's companion.* Princeton University Press.

Athey, Susan, and Guido W. Imbens. 2022. "Design-based Analysis in Difference-In-Differences Settings with Staggered Adoption." *Journal of Econometrics* 226:62–79.

Averett, Susan L, Julie K Smith, and Yang Wang. 2019. "Medicaid expansion and opioid deaths." *Health Economics* 28:1491–1496.

Bartik, Timothy J. 1991. *Who Benefits from State and Local Economic Development Policies?* W. E. Upjohn Institute for Employment Research.

Berry, Steven, James Levinsohn, and Ariel Pakes. 1999. "Voluntary Export Restraints on Automobiles: Evaluating a Trade Policy." *American Economic Review* 89:400–430.

Black, Bernard, Alex Hollingworth, Leticia Nunes, and Kosali Simon. 2019. "The Effect of Health Insurance on Mortality: Power Analysis and What We Can Learn from the Affordable Care Act Coverage Expansions." *NBER Working Paper 25568.*

Borusyak, Kirill, and Peter Hull. 2021. "Non-Random Exposure to Exogenous Shocks: Theory and Applications." *Mimeo.*

———. 2023. "Non-Random Exposure to Exogenous Shocks." *Econometrica* 91:2155–2185.

Borusyak, Kirill, Peter Hull, and Xavier Jaravel. 2022. "Quasi-Experimental Shift-Share Research Designs." *Review of Economic Studies* 89:181–213.

———. 2025a. "A practical guide to shift-share instruments." *Journal of Economic Perspectives* 39:181–204.

———. 2025b. "Design-based identification with formula instruments: A review." *Econometrics Journal.*

Boustan, Leah, Fernando Ferreira, Hernan Winkler, and Eric M. Zolt. 2013. "The effect of rising income inequality on taxation and public expenditures: Evidence from U.S. Municipalities and school districts, 1970-2000." *Review of Economics and Statistics* 95:1291–1302.

Brown, David W., Amanda E. Kowalski, and Ithai Z. Lurie. 2018. "Long-Term Impacts of Childhood Medicaid Expansions on Outcomes in Adulthood." *NBER Working Paper No. 20835.*

Chamberlain, Gary. 1987. "Asymptotic efficiency in estimation with conditional moment restrictions." *Journal of Econometrics* 34:305–334.

———. 1992. "Efficiency Bounds for Semiparametric Regression." *Econometrica* 60:567–596.

Chernozhukov, Victor, Denis Chetverikov, Mert Demirer, Esther Duflo, Christian Hansen, Whitney Newey, and James Robins. 2018. "Double/debiased machine learning for treatment and structural parameters." *Econometrics Journal* 21:C1–C68.

Cohodes, Sarah R., Daniel S. Grossman, Samuel A. Kleiner, and Michael F. Lovenheim. 2016. "The Effect of Child Health Insurance Access on Schooling: Evidence from Public Insurance Expansions." *Journal of Human Resources* 51:727–759.

Conlon, Christopher, and Jeff Gortmaker. 2020. "Best practices for differentiated products demand estimation with PyBLP." *RAND Journal of Economics* 51:1108–1161.

Courtemanche, Charles, James Marton, Benjamin Ukert, Aaron Yelowitz, and Daniela Zapata. 2017. "Early impacts of the Affordable Care Act on health insurance coverage in Medicaid expansion and non-expansion states." *Journal of Policy Analysis and Management* 36:178–210.

Coussens, Stephen, and Jann Spiess. 2021. "Improving Inference from Simple Instruments Through Compliance Estimation." *Mimeo.*

Cullen, Julie Berry, and Jonathan Gruber. 2000. "Does Unemployment Insurance Crowd out Spousal Labor Supply?" *Journal of Labor Economics* 18:546–572.

Currie, Janet, and Jonathan Gruber. 1996a. "Health Insurance Eligibility, Utilization of Medical Care, and Child Health." *The Quarterly Journal of Economics* 111:431–466.

———. 1996b. "Saving Babies: The Efficacy and Cost of Recent Changes in the Medicaid Eligibility of Pregnant Women." *Journal of Political Economy* 104:1263–1296.

Cutler, David M., and Jonathan Gruber. 1996. "Does Public Insurance Crowd out Private Insurance?" *The Quarterly Journal of Economics* 111:391–430.

East, Chloe N., and Elira Kuka. 2015. "Reexamining the consumption smoothing benefits of Unemployment Insurance." *Journal of Public Economics* 132:32–50.

Fiedler, Miroslav. 2003. "Remarks on the Sherman-Morrison-Woodbury formulae" [in en]. *Mathematica Bohemica* 128:253–262. Accessed October 18, 2025.

Frean, Molly, Jonathan Gruber, and Benjamin D. Sommers. 2017. "Premium subsidies, the mandate, and Medicaid expansion: Coverage effects of the Affordable Care Act." *Journal of Health Economics* 53:72–86.

Gao, Mengsi. 2024. "Endogenous interference in randomized experiments." *arXiv preprint arXiv:2412.02183.*

Goldsmith-Pinkham, Paul, Isaac Sorkin, and Henry Swift. 2020. "Bartik Instruments : What, When, Why, and How." *American Economic Review* 110:2586–2624.

Gruber, Jonathan. 2003. "Medicaid." In *Means-tested transfer programs in the United States,* 15–78. University of Chicago Press.

Hackmann, Martin B. 2019. "Incentivizing Better Quality of Care: The Role of Medicaid and Competition in the Nursing Home Industry." *American Economic Review* 109:1684–1716.

Hansen, Lars Peter. 1985. "A method for calculating bounds on the asymptotic covariance matrices of generalized method of moments estimators." *Journal of Econometrics* 30:203–238.

Imbens, Guido W., and Joshua D. Angrist. 1994. "Identification and Estimation of Local Average Treatment Effects." *Econometrica* 62:467–475.

Kaestner, Robert, Bowen Garrett, Jiajia Chen, Anuj Gangopadhyaya, and Caitlyn Fleming. 2017. "Effects of ACA Medicaid Expansions on Health Insurance Coverage and Labor Supply." *Journal of Policy Analysis and Management* 36:608–642.

Kline, Patrick, and Andres Santos. 2012. "A Score Based Approach to Wild Bootstrap Inference." *Journal of Econometric Methods* 1:23–41.

Leung, Pauline, and Alexandre Mas. 2018. "Employment Effects of the ACA Medicaid Expansions." *Industrial Relations* 57:206–234.

Maclean, Johanna Catherine, and Brendan Saloner. 2019. "The Effect of Public Insurance Expansions on Substance Use Disorder Treatment: Evidence from the Affordable Care Act." *Journal of Policy Analysis and Management* 38:366–393.

Miller, Sarah, and Laura R Wherry. 2017. "Health and access to care during the first 2 years of the ACA Medicaid expansions." *New England Journal of Medicine* 376:947–956.

Newey, Whitney K. 1990. "Semiparametric efficiency bounds." *Journal of applied econometrics* 5:99–135.

Newey, Whitney K, and Daniel McFadden. 1994. "Large sample estimation and hypothesis testing." *Handbook of econometrics* 4:2111–2245.

Newey, Whitney K. 1989. "The Asymptotic Variance of Semiparametric Estimators." *Unpublished manuscript.*

Poirier, Alexandre. 2017. "Efficient estimation in models with independence restrictions." *Journal of Econometrics* 196:1–22.

Robinson, P.M. 1988. "Root-N-Consistent Semiparametric Regression." *Econometrica* 56:931–954.

Ross, Nathan. 2011. “Fundamentals of Stein’s method” [in en]. *Probability Surveys* 8:210–293. Accessed October 27, 2025.

Ruggles, Steven, Sarah Flood, Ronald Goeken, Josiah Grover, Erin Meyer, Jose Pacas, and Matthew Sobek. 2020. *IPUMS USA: Version 10.0.* Minneapolis, MN.

Wager, Stefan. 2025. “Causal inference: A statistical learning approach.” *URL https://web. stanford. edu/swager/causal_inf_book. pdf.*

Wooldridge, Jeffrey M. 2002. “Econometric analysis of cross section and panel data MIT press.” *Cambridge, MA.*

A Theory Appendix

A.1 Predetermined Covariates

Here we generalize our theoretical results to allow for an intercept and other predetermined covariates to be included in estimation. Consider:

Assumption A1. *(a) For a symmetric idempotent matrix M which is measurable with respect to w, we have $y = M\check{y}$, $x = M\check{x}$, and $\varepsilon = M\check{\varepsilon}$; (b) $\mathbb{E}\left[\check{\varepsilon}\check{\varepsilon}' \mid g, w\right]$ is almost surely invertible.*

This assumption nests the case considered in the main text (by setting $M = \mathbb{I}$); by the Frisch–Waugh–Lovell theorem it also allows y_i, x_i, and ε_i to be residualized in-sample on some vector of linearly independent covariates $m_i(w)$. The latter corresponds to $M = \mathbb{I} - m(w)\left(m(w)'m(w)\right)^{-1} m(w)'$, with $m(w)$ collecting the $m_i(w)$. In this case, $\mathbb{E}\left[\varepsilon\varepsilon' \mid g, w\right]$ is necessarily rank-deficient which requires restating several results from the main text. To do so, we let $Q^\dagger$ denote the pseudo-inverse of a generic matrix Q and let $\text{Col}(Q)$ denote the column space of Q. We then have:

Proposition A1. *Under Assumptions 1 and A1(a), all recentered instruments satisfy the exogeneity condition (2). Moreover, among instruments in the column space of M, only recentered instruments satisfy (2): for any distribution of (z, g, w) with $z \in \text{Col}(M)$ there exists a conditional distribution of $\varepsilon \mid (z, g, w)$ such that $\varepsilon \in \text{Col}(M)$ and Assumption 1 holds but (2) fails unless z is a deterministic function of (g, w) satisfying (3).*

Proposition 1 is a special case corresponding to $M = \mathbb{I}$. When M is instead reduced-rank, the second statement of Proposition A1 imposes an extra constraint on the adversarial choice of ε: that it has to be in the column space of M. Because of this, the implication is also weaker: for instrument validity, one needs $Mz \in \mathfrak{R}$ which is weaker than $z \in \mathfrak{R}$. However, restricting attention to $z \in \text{Col}(M)$ is without loss since $\hat{\beta}\left[z\right] = \hat{\beta}\left[Mz\right]$ for any z.

We next state generalizations of Theorem 1, Proposition 3, and Lemma 2:

Theorem A1. *Suppose Assumptions 1 and A1 hold. Then the recentered instrument*

$$z^* = \mathbb{E}\left[\varepsilon\varepsilon' \mid g, w\right]^\dagger \left(\mathbb{E}\left[x \mid g, w\right] - \mathbb{E}\left[\mathbb{E}\left[\varepsilon\varepsilon' \mid g, w\right]^\dagger \mid w\right]^\dagger \mathbb{E}\left[\mathbb{E}\left[\varepsilon\varepsilon' \mid g, w\right]^\dagger \mathbb{E}\left[x \mid g, w\right] \mid w\right]\right).$$

solves

$$z^* \in \arg\min_{z \in \mathfrak{R}} \mathcal{V}\left[z\right].$$

Moreover, for $b = \mathbb{E}\left[x \mid g, w\right]$ and $A = \mathbb{E}\left[\varepsilon\varepsilon' \mid g, w\right]$,

$$\mathcal{V}\left[z^*\right] = \mathbb{E}\left[b'A^\dagger b - \mathbb{E}\left[b'A^\dagger \mid w\right]\mathbb{E}\left[A^\dagger \mid w\right]^\dagger \mathbb{E}\left[A^\dagger b \mid w\right]\right]^{-1}.$$

When $\mathbb{E}\left[\varepsilon\varepsilon' \mid g, w\right] = \mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]$ a.s., these simplify to

$$z^* = \mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]^\dagger \left(\mathbb{E}\left[x \mid g, w\right] - \mathbb{E}\left[x \mid w\right]\right) \tag{A1}$$

and

$$\mathcal{V}\left[z^*\right] = \mathbb{E}\left[\left(b - \mathbb{E}\left[b \mid w\right]\right)' \mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]^\dagger \left(b - \mathbb{E}\left[b \mid w\right]\right)\right]^{-1}.$$

Proposition A2. *Suppose Assumption A1 holds with* $\mathrm{Var}\left[\breve{\varepsilon} \mid g, w\right]$ *a.s. invertible, and let $\psi = \mathbb{E}\left[\varepsilon \mid w\right]$ and $\Omega = \mathrm{Var}\left[\varepsilon \mid w\right]$ be the conditional mean and variance of the errors. Then z^* from equation (A1) can be written as*

$$z^* = \Omega^\dagger \left(\tilde{z} - \nu\rho\psi\right),$$

where $\rho\psi = \frac{\psi'\Omega^\dagger \tilde{z}}{\psi'\Omega^\dagger\psi}\psi$ is the $\Omega^\dagger$-weighted projection of $\tilde{z}$ on ψ, and $\nu = \frac{\psi'\Omega^\dagger\psi}{1+\psi'\Omega^\dagger\psi} \in [0, 1)$.

Lemma A1. *Suppose Assumption A1 holds. Let $\bar{\lambda} \geq 0$ be some constant and let $\mathcal{E}$ be the class of random vectors $e = (e_1, \ldots, e_N)'$ such that the maximum eigenvalue of $\mathbb{E}\left[ee' \mid g, w\right]$ is bounded by $\bar{\lambda}$ uniformly over (g, w). Suppose the only knowledge a researcher has about ε is that $\varepsilon = M\breve{\varepsilon}$ for $\breve{\varepsilon} \in \mathcal{E}$. Then $\tilde{z}$ is approximately minimax, in that:*

$$\tilde{z} \in \arg\min_{z \in \mathfrak{R}} \max_{e \in \mathcal{E}} \frac{\mathrm{Var}\left[z'Me\right]}{\mathbb{E}\left[z'x\right]^2}.$$

A.2 Optimal Shift-Share Instruments

Let $\mathfrak{R}^S \subset \mathfrak{R}$ denote the class of recentered shift-share instruments $z = S\left(g - \mathbb{E}\left[g \mid w\right]\right)$ that are characterized by an $N \times K$ matrix S that is measurable with respect to w.

Proposition A3. *Suppose Assumptions 1 and A1 hold, $\mathbb{E}\left[\varepsilon\varepsilon' \mid g, w\right] = \mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]$ a.s., and* $\mathrm{Var}\left[g \mid w\right]$ *is a.s. invertible. Let $\tilde{g} = g - \mathbb{E}\left[g \mid w\right]$. Consider the recentered shift-share instrument*

$$z^S = \mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]^\dagger \tilde{z}^S \quad \text{for } \tilde{z}^S = \mathbb{E}\left[x\tilde{g}' \mid w\right] \mathrm{Var}\left[\tilde{g} \mid w\right]^{-1} \tilde{g}.$$

The associated $\hat{\beta}\left[z^S\right]$ has the smallest approximate variance of all recentered shift-share IV esti-

mators

$$z^S \in \arg\min_{z \in \mathfrak{R}^S} \mathcal{V}[z].$$

A.3 Reaching the Semi-Parametric Efficiency Bound

Proposition A4. *Suppose the observations of $(y_i, x_i, \varepsilon_i, g_i, w_i)$ are* iid *across i (with distributions not changing with N) and that Assumptions 1 and 2 hold with $w = (w_i)_{i=1}^N$ and $g = (g_i)_{i=1}^N$. Suppose further $\mathbb{E}\left[\varepsilon_i^2 \mid g_i, w_i\right] = \mathbb{E}\left[\varepsilon_i^2 \mid w_i\right]$ as in Theorem 1. Then if $\hat{\beta}[z^*]$ is regular and the sequence $\frac{\left(\tilde{z}'\Omega^{-1}\psi\right)^2}{1+\psi'\Omega^{-1}\psi}$ is uniformly integrable, $\hat{\beta}[z^*]$ asymptotically achieves the Chamberlain (1992) semi-parametric efficiency bound.*

A.4 Conditions for Recentered IV Regularity

A full analysis of the asymptotic normality of recentered IV estimators and asymptotic inference on them—an open problem as pointed out by Borusyak et al. (2025b)—is beyond the scope of the current paper. The special case of shift-share instruments has been worked out by Adão et al. (2019, Proposition 4), assuming a growing number of shocks and sufficiently dispersed shock exposure. It is clear their argument can be extended to the case where the instrument can be well approximated by a shift-share construction. Building on one of the consistency results of Borusyak and Hull (2023, Proposition 5), here we provide a complementary set of conditions for regularity considering the case where the recentered instrument is defined by an arbitrary nonlinear formula but depends only on a small set of mutually independent shocks, with non-overlapping sets of shocks for most observation pairs. We impose:

Assumption A2. *(a) $z_i = f_i(g_{A_i}; w) \in \mathfrak{R}$ where $g_{A_i} \equiv \{g_k : k \in A_i\}$ for $A_i \equiv A_i(w) \subseteq \{1, \ldots, K\}$;*
(b) For $D_i = \{j = 1, \ldots, N : A_i \cap A_j \neq \emptyset\}$, $|D_i| \leq \bar{D}$ uniformly across i and N;
(c) $g \perp\!\!\!\perp \varepsilon \mid w$, and the components of g are mutually independent conditional on w;
(d) $\frac{1}{N}\sum_i z_i x_i \xrightarrow{p} \Pi \neq 0$;
(e) $V_N \equiv \frac{1}{N}\mathrm{Var}\left[\sum_i z_i \varepsilon_i \mid \varepsilon, w\right] \xrightarrow{p} V > 0$;
(f) $\mathbb{E}\left[z_i^4 \mid w\right] \leq B_z^4$, and $\frac{1}{N}\sum_i \varepsilon_i^4 = O_p(1)$.

Part (a) of Assumption A2 defines the set of shocks the formula instrument depends on while part (b) imposes sparsity of the bipartite graph of dependencies between observations and shocks. The latter implies a growing number of shocks, $K \to \infty$. Part (c) imposes a stronger version of Assumption 1 as well as mutual independence of the shocks. Part (d) imposes a standard relevance condition, which holds if the first-stage relationship between x_i and z_i is strong and if z_i does not concentrate as $N \to \infty$. Part (e) requires the conditional variance of $\frac{1}{\sqrt{N}}\sum z_i \varepsilon_i$ to stabilize. Parts (b), (d), and (e) naturally hold together when the number of shocks K, observations N, and shock-observation links $L = \sum_i |A_i|$ are of the same order (i.e. $K/N \to \kappa$ and $L/N \to \ell$ for $0 < \kappa, \ell < \infty$) and when links are dispersed across observations and shocks. Part (f) imposes regularity conditions.

Leveraging Stein's method for proving asymptotic normality with locally dependent random variables (specifically as in Ross (2011)), we have:

Proposition A5. *Suppose Assumption A2 holds. Then the recentered estimator $\hat{\beta}[z]$ is regular with*

$$\sqrt{N}\left(\hat{\beta}[z]-\beta\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, V/\Pi^2).$$

We conjecture that most parts of Assumption A2 can be relaxed, in particular allowing $\max_i |D_i|$ to grow slowly with N. Intuitively, consider the following scenario: each observation is still exposed to a small number of shocks, i.e., $\max_i |A_i|$ is bounded, which implies that L is of the same order as N. We maintain that shock exposure is dispersed, which now means that the largest number of observations exposed to a given shock is of the same order as the average, $O(L/K)$. However, suppose L/K grows sufficiently slowly, with $\max_i |D_i|$ of the same order. The proof of Proposition A5 extends with no change if part (b) of Assumption A2 is relaxed to have $\max_i |D_i| = o(N^{1/4}V_N^{3/4} + N^{1/3}V_N^{2/3})$. Part (d) may also expected to hold as z_i, which depends only on a fixed number of shocks, need not concentrate. Key to the asymptotic normality of $\hat{\beta}[z]$ is then the behavior of $\mathrm{Var}\left[\sum_i z_i\varepsilon_i \mid \varepsilon, w\right]$ in part (e). Similar to the analysis in Gao (2024), different assumptions on mutual dependence of $z_i\varepsilon_i$ among observations exposed to common shocks may lead to regular estimators with different rates. We leave a formal analysis of this scenario to future work.

A.5 Plug-In Estimation of the Optimal Instrument

Let $\mathbb{E}[\varepsilon \mid g, w] = \mathbb{E}[\varepsilon \mid w] = \psi$, $\mathrm{Var}[\varepsilon \mid g, w] = \mathrm{Var}[\varepsilon \mid w] = \Omega$, and $D \equiv \mathbb{E}[\varepsilon\varepsilon' \mid g, w] = \Omega + \psi\psi'$. Also let $\mathbb{E}[x \mid g, w] = p$, $\mathbb{E}[x \mid w] = \mathbb{E}[p \mid w] = \mu$, and $h = p - \mu$. Suppose we have estimates $\hat{\Omega}$ (assumed symmetric and positive semi-definite), $\hat{\psi}$, and $\hat{p}$ obtained with cross-fitting; i.e., with the relevant functions estimated from an independent sample. Set $\hat{D} = \hat{\Omega} + \hat{\psi}\hat{\psi}'$, $\hat{\mu} = \mathbb{E}[\hat{p} \mid w]$ (computed exactly by integrating over the $g \mid w$ distribution), and $\hat{h} = \hat{p} - \hat{\mu}$. We will condition on the sample used for cross-fitting throughout, treating the estimated functions as fixed and thus $\hat{D}, \hat{\Omega}, \hat{\psi}, \hat{\mu}$ as measurable with respect to w, and $\hat{p}, \hat{h}$ measurable with respect to g, w. The estimates from multiple folds can be combined using standard DML1 and DML2 arguments (Chernozhukov et al. 2018; Wager 2025). We also define $\delta = \hat{h} - h$ and $\tilde{\varepsilon} = \varepsilon - \psi$, with $\mathbb{E}[\delta \mid w] = 0$, $\mathbb{E}[\tilde{\varepsilon} \mid g, w] = 0$, and $\mathbb{E}[\tilde{\varepsilon}\tilde{\varepsilon}' \mid g, w] = \mathbb{E}[\tilde{\varepsilon}\tilde{\varepsilon}' \mid w] = \Omega$. We write λ_{min} for the smallest eigenvalue and $\|\cdot\|$ and $\|\cdot\|_{\mathrm{op}}$ for the L^2-norm and operator norm, respectively. Throughout we omit the phrase "almost surely." All random variables are implicitly indexed by N; we omit this dependence for notational ease.

We consider the case where the estimator $\hat{\beta}[z^*] = z^{*\prime}y/z^{*\prime}x$ corresponding to the oracle instrument $z^* = D^{-1}h$, is regular, i.e. that the first stage $\frac{1}{N}z^{*\prime}x \xrightarrow{p} \Pi \neq 0$ and, for some sequence $r_N \to \infty$, $\frac{r_N}{N}z^{*\prime}\varepsilon$ converges to a nondegenerate distribution. We are interested in whether the feasible estimator $\hat{\beta}[\hat{z}^*] = \hat{z}^{*\prime}y/\hat{z}^{*\prime}x$ corresponding to the instrument $\hat{z}^* = \hat{D}^{-1}\hat{h}$ has the same asymptotic distribution. We impose:

Assumption A3. *(a)* $\sqrt{\frac{1}{N}\sum_i\left(\hat{h}_i - h_i\right)^2} \xrightarrow{p} 0$, $\sqrt{\frac{1}{N}\sum_i\left(\hat{\psi}_i - \psi_i\right)^2} \xrightarrow{p} 0$, *and* $\left\|\hat{\Omega}^{-1} - \Omega^{-1}\right\|_{op} \xrightarrow{p} 0$;

(b) There exists $c > 0$ such that $\lambda_{min}(\Omega) \geq c$ and $\lambda_{min}\left(\hat{\Omega}\right) \geq c$ with probability approaching 1;

(c) The following are $O_p(1)$: $\|\mathrm{Var}[\varepsilon \mid w]\|_{op}$, $\|\mathrm{Var}[h \mid w]\|_{op}$, $\|\mathrm{Var}[\delta \mid w]\|_{op}$, and $\frac{1}{\sqrt{N}}\|x\|$;

(d) $r_N = O(\sqrt{N})$.

Assumption A3 gives regularity conditions. Part (a) says says that $\hat{h}$, $\hat{\psi}$, and $\hat{\Omega}^{-1}$ are consistent in the appropriate sense. Part (b) requires the eigenvalues of Ω and $\hat{\Omega}$ to be bounded away from zero, yielding stability of their inverses. Part (c) requires that ε and h are sufficiently mutually independent, that δ is either sufficiently small or sufficiently mutually independent, and a mild regularity condition on x. Part (d) precludes unusually fast convergence rates of the oracle recentered IV estimator. Then we have:

Proposition A6. *Suppose Assumption A3 holds. Then if $\hat{\beta}\,[z^*]$ is regular, $\hat{\beta}\,[\hat{z}^*]$ has the same asymptotic distribution.*

Cross-fitting and the fact that the instrument is recentered are key to the proof of Proposition A6. It is also important that, when we set $\hat{D} = \hat{\Omega} + \hat{\psi}\hat{\psi}'$, we are effectively using a Neyman-orthogonal moment for estimation because—by the Sherman–Morrison formula—the estimator is effectively controlling for $\hat{\psi}$ (up to the regularization; see Proposition 3).

B Empirical Appendix

B.1 Data Construction

Our application uses a repeated cross-section of annual data from the American Community Survey (ACS; Ruggles et al. 2020). Our main estimates use representative 1% samples from 2013 and 2014; we use an analogous sample from 2012 to estimate pre-trends. We restrict these samples to non-disabled adults (aged 21-64) residing in one of the 43 states eligible for Medicaid expansion under the ACA. To define this sample of states we follow Frean et al. (2017) in excluding "early expansion" states which expanded Medicaid after the ACA but before 2013, as well as Massachusetts and Vermont which had made all adults with household income less than 138% FPL eligible prior to the ACA. We also follow Frean et al. (2017) in designating 19 of these states as having expanded under the ACA in 2014, with 24 not expanding.[39]

In each year, we classify an individual as insured under Medicaid when they report being covered by Medicaid or an equivalent government-assistance program, excluding Medicare and Veterans Affairs insurance. We classify an individual as having private insurance when she is covered by a plan purchased through an employer or union, or when she purchases this private coverage directly. We further separate individuals covered by employer-sponsored insurance.

We use the formulas in Frean et al. (2017) to define Medicaid eligibility. Income is given by a household's total pre-tax personal income or losses (*inctot*), adjusted for inflation. Other inputs to the eligibility calculation include an indicator for whether an individual is a parent (i.e. an adult with children in the household) and an indicator for whether an individual is in the labor force (*labforce*). We note that these may be noisy proxies for the characteristics actual used to assign Medicaid eligibility.

[39] Frean et al. (2017) study coverage effects over 2014-2015, designating 24 states as having expanded during this time, 21 states as having not expanded, and 6 states as expanding early. We use their classification system as of 2014, when only 19 of their 24 states have expanded, and additionally exclude Massachusetts and Vermont.

Our simulated eligibility instrument is constructed by simulating the average Medicaid eligibility of a representative 10% sample of our analysis data under different state policies. Namely we use the representative sample to simulate two shares: that of individuals who would be eligible had their state expanded eligibility in 2014 to everyone under 138% of FPL (24.5%), and that of individuals who would be eligible if their state kept 2013 policy intact (11.6%). We assign the difference in these shares to all individuals in 2014 residing in expansion states and zero to all other individuals.

Our recentered instrument is constructed by predicting the actual Medicaid eligibility of each individual. In 2013 we use actual 2013 eligibility policies, again following Frean et al. (2017). In 2014 we predict eligibility by combining information on the 2013 policies and a state's decision to expand. An individual is eligible for Medicaid in 2014 if either she was eligible under the 2013 policies of her state (whether or not the state expanded eligibility) or if her household income is below 138% FPL and her state expanded eligibility under the ACA. To compute the expected instrument μ_i we first identify individuals who would have been eligible in 2014 if their state expanded but not otherwise (the "exposed sample"). Outside of this sample the expected instrument in 2014 is simply the individual's actual 2014 eligibility, while inside this sample the expected instrument in 2014 is the fraction of states which expanded conditional on the governor's party. The 2013 expected instrument is actual 2013 eligibility for all individuals. Political party affiliation of state governors is determined as of December 2013. In all regressions we control for a Republican governor indicator interacted with year. In robustness checks we control for other time-interacted state characteristics: a state's 2012 median income or share insured under Medicaid (both from the ACS).

B.2 Power Simulations

We verify large and robust power gains from recentered IV in a Monte Carlo study, in which the true causal effect and the shock assignment process are known. We draw 999 counterfactual state expansion decisions by choosing random sets of 8 Republican- and 11 Democratic-controlled states as expansion states and use these shocks to generate simulated instruments and recentered best predictor instruments. We do not specify a model for e_k^{Δ}, and instead take the best predictor instrument as the endogenous variable. Finally, for the Medicaid take-up and employer-sponsored insurance crowd-out outcomes we take the second-stage residuals from Columns 2 and 6 of Table 2, panel A. These outcomes are unrelated to the endogenous variable by design, corresponding to the true causal effect of zero for all individuals, while keeping the correlation structure from the actual data. With these generated data, we re-estimate the simulated and recentered IV specifications as in our baseline implementation in Panel A of Table 2. By design, both sets of estimates should be centered at the true effects of zero, while we expect the recentered IV procedure to systematically reject a larger set of alternative hypotheses.

Appendix Figure A2 shows the distribution of simulated and recentered IV estimates for the two outcomes across Monte Carlo draws. Both estimators are approximately unbiased, with the distributions in both panels centered around the true effect of zero. However, consistent with the much shorter confidence intervals in Table 2, the distribution of recentered IV coefficients is dramatically tighter around this mean. The estimator standard deviation falls from 0.014 to 0.006 as we move from the simulated IV to recentered IV in Panel A, with a larger decline from 0.020 to 0.007 in Panel B. With minimal bias, these correspond to simulated root mean-squared error reductions of 58.5% and 66.5% with the recentered IV, respectively.

Figure 1 shows that these reductions in estimator variance translate to increased rejection rates of false null hypotheses for both outcomes, while also suggesting the wild bootstrap 95% confidence intervals in Table 2 have approximately correct size. Away from the true null hypothesis of zero the recentered IV power curve is much more steeply sloping, with uniformly higher rejection rates. With the Medicaid take-up outcome, for example, the recentered IV is found to reject coefficients outside the range of $[-0.018, 0.017]$ with probability of at least 0.8, while the simulated IV only has such high power outside a nearly three times as long range, of $[-0.042, 0.056]$. For the employer-sponsored insurance crowd-out outcome this contrast in minimum detectable effects is even starker, at $[-0.022, 0.018]$ for the recentered IV versus $[-0.073, 0.051]$ for the simulated IV.

C Proofs

C.1 Proof of Proposition A1

The first statement of the propositions follows by the law of iterated expectations, similarly to Borusyak and Hull (2023) except with a slightly weaker Assumption 1: for a recentered formula instrument z_i,

$$\mathbb{E}\left[\frac{1}{N}\sum_i z_i\varepsilon_i\right] = \frac{1}{N}\sum_i \mathbb{E}\left[z_i\mathbb{E}\left[\varepsilon_i \mid g, w\right]\right] = \frac{1}{N}\sum_i \mathbb{E}\left[z_i\mathbb{E}\left[\varepsilon_i \mid w\right]\right] = \frac{1}{N}\sum_i \mathbb{E}\left[\mathbb{E}\left[z_i \mid w\right]\mathbb{E}\left[\varepsilon_i \mid w\right]\right] = 0.$$

We now prove the second statement of the general Proposition A1, focusing for clarity on the case of discrete g, w, and z. The continuously-distributed case follows similarly under appropriate regularity conditions.

To show that, if $z \in \text{Col}(M)$, z has to be a formula instrument (i.e. measurable with respect to g, w), suppose by contradiction there is an observation ι and values $(\bar{g}, \bar{w})$ in the support of (g, w) such that z_ι takes distinct values $\bar{z}_1$ and $\bar{z}_2$ with probabilities $\pi_{\bar{z}_j|\bar{g},\bar{w}} > 0$, $j = 1, 2$. Consider the following data-generating process for ε: $\varepsilon = M\check{\varepsilon}$ where $\check{\varepsilon}_i = 0$ for $i \neq \iota$ and

$$\check{\varepsilon}_\iota = \begin{cases} (\bar{z}_1 - \bar{z}_2)/\pi_{\bar{z}_1|\bar{g},\bar{w}} & g = \bar{g}, w = \bar{w}, z_\iota = \bar{z}_1 \\ -(\bar{z}_1 - \bar{z}_2)/\pi_{\bar{z}_2|\bar{g},\bar{w}} & g = \bar{g}, w = \bar{w}, z_\iota = \bar{z}_2 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\varepsilon \in \text{Col}(M)$. It is also straightforward to verify that this ε satisfies $0 = \mathbb{E}\left[\varepsilon \mid g, w\right] =$

$\mathbb{E}[\varepsilon \mid w]$. Yet,

$$\begin{aligned}\mathbb{E}\left[z'\varepsilon\right] &= \mathbb{E}\left[z'M\check{\varepsilon}\right] \\ &= \mathbb{E}\left[z'\check{\varepsilon}\right] \\ &= \left(\bar{z}_1 \cdot (\bar{z}_1 - \bar{z}_2) - \bar{z}_2 \cdot (\bar{z}_1 - \bar{z}_2)\right) \cdot Pr\left(g = \bar{g}, w = \bar{w}\right) \\ &= \left(\bar{z}_1 - \bar{z}_2\right)^2 \cdot Pr\left(g = \bar{g}, w = \bar{w}\right) \neq 0,\end{aligned}$$

violating (2), where the second line holds because $Mz = z$ for idempotent M and $z \in \text{Col}(M)$.

It remains to be shown that z must be a recentered instrument. Consider a different data-generating process for ε, in which $\mathbb{E}[\varepsilon \mid w] = \mathbb{E}[z \mid w]$ (and again $\varepsilon \in \text{Col}(M)$ since $z \in \text{Col}(M)$). Then, using the fact that z is a formula instrument,

$$\mathbb{E}\left[z'\varepsilon\right] = \mathbb{E}\left[z'\mathbb{E}[\varepsilon \mid g, w]\right] = \mathbb{E}\left[z'\mathbb{E}[\varepsilon \mid w]\right] = \mathbb{E}\left[\mathbb{E}[z \mid w]'\,\mathbb{E}[\varepsilon \mid w]\right] = \mathbb{E}\left[\mathbb{E}[z \mid w]'\,\mathbb{E}[z \mid w]\right] \neq 0,$$

unless $\mathbb{E}[z \mid w] = 0$ a.s., violating (2) again.

C.2 Proof of Proposition 2

Uniform integrability of $\frac{1}{N}z'x$ implies $\mathbb{E}\left[\frac{1}{N}z'x\right] \to \Pi$. Then, denoting the asymptotic distribution of $r_N\left(\hat{\beta}[z] - \beta\right)$ by $\tilde{\mathcal{D}}$ and using the continuous mapping theorem,

$$r_N \frac{\frac{1}{N}z'\varepsilon}{\mathbb{E}\left[\frac{1}{N}z'x\right]} = r_N\left(\hat{\beta}[z] - \beta\right) \cdot \frac{\frac{1}{N}z'x}{\Pi} \cdot \frac{\Pi}{\mathbb{E}\left[\frac{1}{N}z'x\right]} \Rightarrow \tilde{\mathcal{D}},$$

as $\frac{\frac{1}{N}z'x}{\Pi} \xrightarrow{p} 1$, and $\frac{\Pi}{\mathbb{E}\left[\frac{1}{N}z'x\right]} \to 1$. Furthermore, uniform integrability of $\left(r_N\frac{1}{N}z'\varepsilon\right)^2$ implies

$$\text{Var}\left[r_N \frac{\frac{1}{N}z'\varepsilon}{\mathbb{E}\left[\frac{1}{N}z'x\right]}\right] = r_N^2 \mathcal{V}[z] \to V,$$

which is equivalent to the Proposition's claim.

C.3 Proof of Theorem A1

We prove Theorem A1; Theorem 1 is a special case with $M(w) = \mathbb{I}$ and $\mathbb{E}[\varepsilon\varepsilon' \mid g, w] = \mathbb{E}[\varepsilon\varepsilon' \mid w]$.

First, we confirm that $z^* \in \mathfrak{R}$. By properties of the pseudo-inverse for the symmetric matrix $A = M\mathbb{E}[\check{\varepsilon}\check{\varepsilon}' \mid g, w]\,M$, we have $\text{Col}(M) = \text{Col}(A) = \text{Col}(A^\dagger) = \text{Col}\left(\mathbb{E}\left[A^\dagger \mid w\right]\right)$ a.s.. Moreover, $A^\dagger b \in \text{Col}(M)$ a.s. and so $\mathbb{E}\left[A^\dagger b \mid w\right] \in \text{Col}(M)$. Thus,

$$\mathbb{E}[z^* \mid w] = \mathbb{E}\left[A^\dagger b \mid w\right] - \mathbb{E}\left[A^\dagger \mid w\right]\mathbb{E}\left[A^\dagger \mid w\right]^\dagger \mathbb{E}\left[A^\dagger b \mid w\right] = 0.$$

Next, we show that $\mathbb{E}[z'\varepsilon\varepsilon' z^*] = \mathbb{E}[z'x]$ for any $z \in \mathfrak{R}$. Since both x and ε are in $\text{Col}(M)$ and M is symmetric idempotent, we can restrict attention to $z \in \text{Col}(M)$ without loss of generality: z

and Mz produce the same IV estimator. Then:

$$\begin{aligned}
\mathbb{E}\left[z'\varepsilon\varepsilon' z^*\right] &= \mathbb{E}\left[z' A z^*\right] \\
&= \mathbb{E}\left[z' A A^\dagger \left(b - \mathbb{E}\left[A^\dagger \mid w\right]^\dagger \mathbb{E}\left[A^\dagger b \mid w\right]\right)\right] \\
&= \mathbb{E}\left[z' b\right] - \mathbb{E}\left[z' \mathbb{E}\left[A^\dagger \mid w\right]^\dagger \mathbb{E}\left[A^\dagger b \mid w\right]\right] \\
&= \mathbb{E}\left[z' x\right] - 0,
\end{aligned}$$

where the third equality follows because z and $\mathbb{E}\left[A^\dagger \mid w\right]^\dagger$ are in $\mathrm{Col}(M)$, and the last equality follows by the law of iterated expectations and because z is mean-zero conditional on any function of w. Note that $z^* \in \mathrm{Col}(M)$ a.s. and $\mathbb{E}\left[z'\varepsilon\varepsilon' z^*\right] = \mathbb{E}\left[z' x\right]$ also implies $\mathcal{V}\left[z^*\right] = \mathbb{E}\left[z^{*\prime} x\right]^{-1}$.

Following the proof of Theorem 5.3 in Newey and McFadden (1994), let

$$U = \frac{z'\varepsilon}{\mathbb{E}\left[z'x\right]} - \frac{z^{*\prime}\varepsilon}{\mathbb{E}\left[z^{*\prime}x\right]}.$$

Then

$$\begin{aligned}
\mathbb{E}\left[U^2\right] &= \frac{\mathrm{Var}\left[z'\varepsilon\right]}{\mathbb{E}\left[z'x\right]^2} - 2\frac{\mathbb{E}\left[z'\varepsilon\varepsilon' z^*\right]}{\mathbb{E}\left[z'x\right]\mathbb{E}\left[z^{*\prime}x\right]} + \frac{\mathbb{E}\left[z^{*\prime}\varepsilon\varepsilon' z^*\right]}{\mathbb{E}\left[z^{*\prime}x\right]^2} \\
&= \frac{\mathrm{Var}\left[z'\varepsilon\right]}{\mathbb{E}\left[z'x\right]^2} - \frac{1}{\mathbb{E}\left[z^{*\prime}x\right]} \\
&= \mathcal{V}\left[z\right] - \mathcal{V}\left[z^*\right] \geq 0,
\end{aligned}$$

showing that z^* minimizes the approximate estimator variance.

It remains to obtain the expression for $\mathcal{V}\left[z^*\right]$. Repeatedly applying the law of iterated expectations, we have:

$$\begin{aligned}
\mathcal{V}\left[z^*\right] &= \mathbb{E}\left[z^{*\prime}x\right]^{-1} \\
&= \mathbb{E}\left[\left(b - \mathbb{E}\left[A^\dagger \mid w\right]^\dagger \mathbb{E}\left[A^\dagger b \mid w\right]\right)' A^\dagger x\right]^{-1} \\
&= \mathbb{E}\left[\left(b - \mathbb{E}\left[A^\dagger \mid w\right]^\dagger \mathbb{E}\left[A^\dagger b \mid w\right]\right)' A^\dagger b\right]^{-1} \\
&= \mathbb{E}\left[b' A^\dagger b - \mathbb{E}\left[b' A^\dagger \mid w\right] \mathbb{E}\left[A^\dagger \mid w\right]^\dagger A^\dagger b\right]^{-1} \\
&= \mathbb{E}\left[b' A^\dagger b - \mathbb{E}\left[b' A^\dagger \mid w\right] \mathbb{E}\left[A^\dagger \mid w\right]^\dagger \mathbb{E}\left[A^\dagger b \mid w\right]\right]^{-1}.
\end{aligned}$$

In the special case of $A = \mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]$ a.s., we have

$$\begin{aligned} z^* &= A^\dagger \left(\mathbb{E}\left[x \mid g, w\right] - \left(A^\dagger\right)^\dagger A^\dagger \mathbb{E}\left[x \mid w\right] \right) \\ &= \mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]^\dagger \left(\mathbb{E}\left[x \mid g, w\right] - \mathbb{E}\left[x \mid w\right]\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{V}\left[z^*\right] &= \mathbb{E}\left[b' A^\dagger b - \mathbb{E}\left[b \mid w\right]' A^\dagger \left(A^\dagger\right)^\dagger A^\dagger \mathbb{E}\left[b \mid w\right]\right]^{-1} \\ &= \mathbb{E}\left[b' A^\dagger b - \mathbb{E}\left[b \mid w\right]' A^\dagger \mathbb{E}\left[b \mid w\right]\right]^{-1} \\ &= \mathbb{E}\left[\left(b - \mathbb{E}\left[b \mid w\right]\right)' A^\dagger \left(b - \mathbb{E}\left[b \mid w\right]\right)\right]^{-1} \end{aligned}$$

where the last line follows because $\mathbb{E}\left[b' A^\dagger \mathbb{E}\left[b \mid w\right]\right] = \mathbb{E}\left[\mathbb{E}\left[b \mid w\right]' A^\dagger \mathbb{E}\left[b \mid w\right]\right]$ by the law of iterated expectations.

C.4 Proof of Proposition A2

We prove Proposition A2; Proposition 3 is again a special case. By the law of total variance, $\mathbb{E}\left[\varepsilon\varepsilon' \mid w\right] = \Omega + \psi\psi'$. Since $\text{Var}\left[\check{\varepsilon} \mid g, w\right]$ is a.s. invertible, $\text{Col}(\Omega) = \text{Col}(M)$. Moreover, $\psi = M\mathbb{E}\left[\check{\varepsilon} \mid w\right] \in \text{Col}(M)$. By the Sherman-Morrison formula and its extension to the case of possibly rank-deficient Ω and rank-one update $\psi\psi'$ that does not affect the rank of the sum (Fiedler 2003, Theorem 2.9),

$$\left(\Omega + \psi\psi'\right)^\dagger = \Omega^\dagger - \Omega^\dagger \psi \frac{\psi' \Omega^\dagger}{1 + \psi' \Omega^\dagger \psi}.$$

Thus:

$$z^* = \Omega^\dagger \left(\tilde{z} - \frac{\psi' \Omega^\dagger \tilde{z}}{1 + \psi' \Omega^\dagger \psi} \psi \right) = \Omega^\dagger \left(\tilde{z} - \rho\nu\psi\right).$$

C.5 Proof of Lemma 1

We have $\mathbb{E}\left[z'x\right] = \mathbb{E}\left[z'\mathbb{E}\left[x \mid g, w\right]\right] = \mathbb{E}\left[z'\tilde{z}\right]$, where the first equality follows from the law of iterated expectations and the second equality holds because $\mathbb{E}\left[z'\mathbb{E}\left[x \mid w\right]\right] = 0$ for any $z \in \mathfrak{R}$. Thus, maximizing $\frac{\mathbb{E}[z'x]^2}{\mathbb{E}[z'z]\mathbb{E}[x'x]}$ is equivalent to maximizing $\frac{\mathbb{E}[z'\tilde{z}]^2}{\mathbb{E}[z'z]\mathbb{E}[\tilde{z}'\tilde{z}]}$. By the Cauchy-Schwarz inequality, this ratio attains its maximum of one at $z = \tilde{z}$.

C.6 Proof of Lemma A1

We prove Lemma A1; Lemma 2 is again a special case. Since $\mathbb{E}\left[z'\varepsilon\right] = 0$ and z is measurable with respect to (g, w),

$$\text{Var}\left[z'\varepsilon\right] = \mathbb{E}\left[z'\varepsilon\varepsilon' z\right] = \mathbb{E}\left[z'\mathbb{E}\left[\varepsilon\varepsilon' \mid g, w\right] z\right] = \mathbb{E}\left[z' M \mathbb{E}\left[\check{\varepsilon}\check{\varepsilon}' \mid g, w\right] M z\right].$$

By standard results in linear algebra,

$$\max_{e\in\mathcal{E}} z'M'\mathbb{E}\left[ee' \mid g,w\right] Mz = \bar{\lambda} z'Mz,$$

which is achieved when $\mathbb{E}\left[ee' \mid g,w\right] = \bar{\lambda}\frac{Mzz'M}{z'Mz}$ a.s. Thus, the minimax problem simplifies to

$$\min_{z\in\mathfrak{R}} \frac{\mathbb{E}\left[z'Mz\right]}{\mathbb{E}\left[z'x\right]^2}.$$

Since $x \in \mathrm{Col}(M)$, it is without loss to consider

$$\min_{\substack{z\in\mathfrak{R},\\ z\in\mathrm{Col}(M)}} \frac{\mathbb{E}\left[z'z\right]}{\mathbb{E}\left[z'x\right]^2}.$$

Minimizing this expression over all $z \in \mathfrak{R}$ is equivalent to $\max_{z\in\mathfrak{R}} \frac{\mathbb{E}[z'x]^2}{\mathbb{E}[z'z]}$ and thus to $\max_{z\in\mathfrak{R}} \frac{\mathbb{E}[z'x]^2}{\mathbb{E}[z'z]\mathbb{E}[x'x]}$. By Lemma 1, $\tilde{z}$ is a solution. Since $\tilde{z} \in \mathrm{Col}(M)$, it is a solution to the original problem too.

C.7 Proof of Appendix Proposition A3

Let $\Lambda = \mathrm{Var}\left[\tilde{g} \mid w\right]$ be the covariance matrix of the shocks and let $S^* = \mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]^\dagger \mathbb{E}\left[x\tilde{g}' \mid w\right]\Lambda^{-1}$. For brevity, we suppress the dependence of S, S^*, and Λ on w.

As with Theorem 1, we follow the proof of Theorem 5.3 in Newey and McFadden (1994). We first show that $\mathbb{E}\left[z'\varepsilon\varepsilon' z^S\right] = \mathbb{E}\left[z'x\right]$ for any $z \in \mathfrak{R}^S$: repeatedly using the law of iterated expectations,

$$\begin{aligned}
\mathbb{E}\left[z'\varepsilon\varepsilon' z^S\right] &= \mathbb{E}\left[\tilde{g}'S'\varepsilon\varepsilon' S^*\tilde{g}\right] \\
&= \mathbb{E}\left[\tilde{g}'S'\mathbb{E}\left[\varepsilon\varepsilon' \mid g,w\right] S^*\tilde{g}\right] \\
&= \mathbb{E}\left[\tilde{g}'S'\mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]\mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]^\dagger \mathbb{E}\left[x\tilde{g}' \mid w\right]\Lambda^{-1}\tilde{g}\right] \\
&= \mathrm{tr}\,\mathbb{E}\left[\mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]\mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]^\dagger \mathbb{E}\left[x\tilde{g}' \mid w\right]\Lambda^{-1}\tilde{g}\tilde{g}'S'\right] \\
&= \mathrm{tr}\,\mathbb{E}\left[\mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]\mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]^\dagger \mathbb{E}\left[x\tilde{g}' \mid w\right] S'\right] \\
&= \mathrm{tr}\,\mathbb{E}\left[\mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]\mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]^\dagger \mathbb{E}\left[xz' \mid w\right]\right] \\
&= \mathrm{tr}\,\mathbb{E}\left[\mathbb{E}\left[xz' \mid w\right]\right] \\
&= \mathrm{tr}\,\mathbb{E}\left[xz'\right] = \mathbb{E}\left[z'x\right],
\end{aligned}$$

where the third line imposed the condition $\mathbb{E}\left[\varepsilon\varepsilon' \mid g,w\right] = \mathbb{E}\left[\varepsilon\varepsilon' \mid w\right]$ from Theorem 1, and the seventh line follows because $x \in \mathrm{Col}(M)$ a.s. implies $\mathbb{E}\left[xz' \mid w\right] \subseteq \mathrm{Col}(M) = \mathrm{Col}(\mathbb{E}\left[\varepsilon\varepsilon' \mid w\right])$. The rest follows identically to Theorem 1, as $\mathcal{V}\left[z\right] - \mathcal{V}\left[z^S\right] = \mathbb{E}\left[U^2\right] \geq 0$ for $U = \frac{z'\varepsilon}{\mathbb{E}[z'x]} - \frac{z^{S\prime}\varepsilon}{\mathbb{E}[z^{S\prime}x]}$.

C.8 Proof of Proposition A4

By Proposition 2, $\mathcal{V}[z^*]$ is the (rescaled) asymptotic variance of the estimator. We have:

$$\begin{aligned}\mathcal{V}[z^*] &= \mathbb{E}\left[\tilde{z}'\left(\Omega+\psi\psi'\right)^{-1}\tilde{z}\right]^{-1} \\ &= \mathbb{E}\left[\tilde{z}'\left(\Omega^{-1}-\Omega^{-1}\psi\frac{\psi'\Omega^{-1}}{1+\psi'\Omega^{-1}\psi}\right)\tilde{z}\right]^{-1} \\ &= \mathbb{E}\left[\tilde{z}'\Omega^{-1}\tilde{z}-\frac{\left(\tilde{z}'\Omega^{-1}\psi\right)^2}{1+\psi'\Omega^{-1}\psi}\right]^{-1},\end{aligned}$$

where the first line follows from Theorem 1, the second line plugs in the result of Proposition 3, and the third line rearranges terms.

We next show that this expression asymptotically coincides with $\mathbb{E}\left[\tilde{z}'\Omega^{-1}\tilde{z}\right]^{-1}$. Note that

$$\mathbb{E}\left[\tilde{z}'\Omega^{-1}\tilde{z}-\frac{\left(\tilde{z}'\Omega^{-1}\psi\right)^2}{1+\psi'\Omega^{-1}\psi}\right]^{-1}-\mathbb{E}\left[\tilde{z}'\Omega^{-1}\tilde{z}\right]^{-1}=\frac{\mathbb{E}\left[\frac{\left(\tilde{z}'\Omega^{-1}\psi\right)^2}{1+\psi'\Omega^{-1}\psi}\right]}{\mathbb{E}\left[\tilde{z}'\Omega^{-1}\tilde{z}\right]\left(\mathbb{E}\left[\tilde{z}'\Omega^{-1}\tilde{z}\right]-\mathbb{E}\left[\frac{\left(\tilde{z}'\Omega^{-1}\psi\right)^2}{1+\psi'\Omega^{-1}\psi}\right]\right)}$$

and, since $\mathrm{Var}\left[\tilde{z}_i \mid w_i\right] \neq 0$ with positive probability by Assumption 2,

$$\begin{aligned}\mathbb{E}\left[\tilde{z}'\Omega^{-1}\tilde{z}\right] &= \sum_i \mathbb{E}\left[\frac{\tilde{z}_i^2}{\mathrm{Var}\left[\varepsilon_i \mid w_i\right]}\right] \\ &= \mathbb{E}\left[\frac{\mathrm{Var}\left[\tilde{z}_i \mid w_i\right]}{\mathrm{Var}\left[\varepsilon_i \mid w_i\right]}\right]\times N \\ &\to \infty,\end{aligned}$$

so long as $\mathbb{E}\left[\frac{\mathrm{Var}[\tilde{z}_i|w_i]}{\mathrm{Var}[\varepsilon_i|w_i]}\right]$ exists. To establish the result, it remain to show that $\mathbb{E}\left[\frac{\left(\tilde{z}'\Omega^{-1}\psi\right)^2}{1+\psi'\Omega^{-1}\psi}\right]$ does not diverge to infinity. This is obvious if $\psi_i = 0$ a.s. Otherwise, $\frac{1}{\sqrt{N}}\tilde{z}'\Omega^{-1}\psi = O_p(1)$ by the central limit theorem while $\frac{1}{N}\left(1+\psi'\Omega^{-1}\psi\right) \xrightarrow{p} \mathbb{E}\left[\psi_i^2/\mathrm{Var}\left[\varepsilon_i \mid w_i\right]\right] > 0$. Thus $\frac{\left(\tilde{z}'\Omega^{-1}\psi\right)^2}{1+\psi'\Omega^{-1}\psi} = O_p(1)$. By uniform integrability of $\frac{\left(\tilde{z}'\Omega^{-1}\psi\right)^2}{1+\psi'\Omega^{-1}\psi}$, its expectation converges to the finite expectation of the limit distribution: specifically, to

$$\frac{\mathbb{E}\left[\mathrm{Var}\left[\tilde{z}_i \mid w_i\right]\psi_i^2/\mathrm{Var}\left[\varepsilon_i \mid w_i\right]^2\right]}{\mathbb{E}\left[\psi_i^2/\mathrm{Var}\left[\varepsilon_i \mid w_i\right]\right]}.$$

Finally, we show that $\mathbb{E}\left[\tilde{z}'\Omega^{-1}\tilde{z}\right]^{-1}$ coincides with the semi-parametric efficiency bound in this setting. Following the notation of Chamberlain (1992), we have a model that satisfies the moment condition $\mathbb{E}\left[\rho(y_i, x_i, g_i, w_i, \beta, h_0(w_i)) \mid g_i, w_i\right] = 0$ where $\rho(y, x, g, w, \beta, \tau) = y - \beta x - \tau$ and $h_0(\bar{w}) = \mathbb{E}\left[\varepsilon_i \mid w_i = \bar{w}\right]$. Chamberlain (1992) shows the efficiency bound is:

$$J_0^{-1} = \mathbb{E}\left[\mathbb{E}\left[D_0'\Sigma_0^{-1}D_0 \mid w_i\right]-\mathbb{E}\left[D_0'\Sigma_0^{-1}H_0 \mid w_i\right]\mathbb{E}\left[H_0'\Sigma_0^{-1}H_0 \mid w_i\right]^{-1}\mathbb{E}\left[H_0'\Sigma_0^{-1}D_0 \mid w_i\right]\right]^{-1}$$

for

$$
\begin{aligned}
D_0 &= \mathbb{E}\left[\frac{\partial}{\partial\beta}\rho(y_i, x_i, g_i, w_i, \beta, h_0(w_i)) \mid g_i, w_i\right] \\
\Sigma_0 &= \mathbb{E}\left[\rho(y_i, x_i, g_i, w_i, \beta, h_0(w_i))\rho(y_i, x_i, g_i, w_i, \beta, h_0(w_i))' \mid g_i, w_i\right] \\
H_0 &= \mathbb{E}\left[\frac{\partial}{\partial\tau}\rho(y_i, x_i, g_i, w_i, \beta, h_0(w_i))' \mid g_i, w_i\right].
\end{aligned}
$$

In our model, $D_0 = -\mathbb{E}[x_i \mid g_i, w_i]$ and $H_0 = -1$. Furthermore $\Sigma_0 = \mathbb{E}\left[(\varepsilon_i - \mathbb{E}[\varepsilon_i \mid w_i])^2 \mid g_i, w_i\right] = \mathrm{Var}[\varepsilon_i \mid w_i]$ since $\mathbb{E}\left[\varepsilon_i^2 \mid g_i, w_i\right] = \mathbb{E}\left[\varepsilon_i^2 \mid w_i\right]$. Hence

$$
\begin{aligned}
J_0 =& \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}[x_i \mid g_i, w_i]' \mathrm{Var}[\varepsilon_i \mid w_i]^{-1} \mathbb{E}[x_i \mid g_i, w_i] \mid w_i\right]\right] \\
&- \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}[x_i \mid g_i, w_i]' \mathrm{Var}[\varepsilon_i \mid w_i]^{-1} \mid w_i\right] \mathbb{E}\left[\mathrm{Var}[\varepsilon_i \mid w_i]^{-1} \mid w_i\right]^{-1} \mathbb{E}\left[\mathrm{Var}[\varepsilon_i \mid w_i]^{-1} \mathbb{E}[x_i \mid g_i, w_i] \mid w_i\right]\right] \\
=& \mathbb{E}\left[\mathbb{E}\left[(\mathbb{E}[x_i \mid g_i, w_i] - \mathbb{E}[x_i \mid w_i])' \mathrm{Var}[\varepsilon_i \mid w_i]^{-1} (\mathbb{E}[x_i \mid g_i, w_i] - \mathbb{E}[x_i \mid w_i]) \mid w_i\right]\right] \\
=& \mathbb{E}\left[\frac{\mathrm{Var}[\tilde{z}_i \mid w_i]}{\mathrm{Var}[\varepsilon_i \mid w_i]}\right],
\end{aligned}
$$

while as shown above the variance of the optimal IV asymptotically coincides with $\mathbb{E}\left[\tilde{z}'\Omega^{-1}\tilde{z}\right]^{-1} = \frac{1}{N}\mathbb{E}\left[\frac{\mathrm{Var}[\tilde{z}_i|w_i]}{\mathrm{Var}[\varepsilon_i|w_i]}\right]^{-1}$, where $\frac{1}{N}$ reflects the notational difference between the asymptotic variance (scaled by N) and the approximate variance (not scaled).

C.9 Proof of Proposition A5

Consider $R_i = z_i\varepsilon_i$ and $R = \sum_i R_i$. Conditional on (ε, w), R_i and R_j are independent if $A_i \cap A_j = \emptyset$. Moreover, $\mathbb{E}\left[R_i^4 \mid \varepsilon, w\right] = \mathbb{E}\left[z_i^4 \mid w_i\right] \cdot \varepsilon_i^4$ and similarly $\mathbb{E}\left[|R_i|^3 \mid \varepsilon, w\right] \le B_z^3 |\varepsilon_i|^3$. Thus, by Theorem 3.6 in Ross (2011), the Wasserstein distance d_W between the distribution of $R/\sqrt{NV_N}$ given (ε, w) and the standard normal distribution is bounded as follows:

$$
d_W \le \frac{\bar{D}^2}{N^{1/2}V_N^{3/2}} B_z^3 \left(\frac{1}{N}\sum_{i=1}^N |\varepsilon_i|^3\right) + \frac{\sqrt{28}\bar{D}^{3/2}}{\sqrt{\pi}N^{1/2}V_N} B_z^2 \left(\frac{1}{N}\sum_{i=1}^N \varepsilon_i^4\right)^{1/2} \xrightarrow{p} 0
$$

because $\frac{1}{N}\sum_{i=1}^N |\varepsilon_i|^3 \le \left(\frac{1}{N}\sum_{i=1}^N \varepsilon_i^4\right)^{3/4}$ by the Hölder inequality. Thus, $R/\sqrt{NV_N} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$ conditionally on (ε, w) and therefore unconditionally. Moreover, since $V_N \xrightarrow{p} V > 0$,

$$
\frac{1}{\sqrt{N}}\sum_i z_i\varepsilon_i \xrightarrow{\mathcal{D}} \mathcal{N}(0, V).
$$

Using that $\sqrt{N}(\hat{\beta}[z] - \beta) = \frac{1}{\sqrt{N}}\sum_i z_i\varepsilon_i / \frac{1}{N}\sum_i z_i x_i$ and that the denominator converges in probability to Π concludes the proof.

C.10 Proof of Proposition A6

Before proving the proposition, we state and prove four useful lemmas.

Lemma C1. *Let random variable V be measurable with respect to the information set $\mathcal{F}$ and suppose $\mathbb{E}\left[U^2 \mid \mathcal{F}\right] \le V$. Then $U = O_p(1)$ if $V = O_p(1)$, and $U = o_p(1)$ if $V = o_p(1)$.*

Proof. If $V = O_p(1)$, fix $\epsilon > 0$ and choose $K < \infty$ and N_0 such that $Pr\left(V > K\right) < \epsilon/2$ for all $N \ge N_0$. Then for any $M > 0$,

$$Pr\left(|U| > M\right) \le Pr\left(V > K\right) + Pr\left(|U| > M, V \le K\right).$$

For the second term, condition on $\mathcal{F}$ and use conditional Markov:

$$\begin{aligned} Pr\left(|U| > M, V \le K\right) &= \mathbb{E}\left[\mathbf{1}\left[V \le K\right] \cdot P\left(|U| > M \mid \mathcal{F}\right)\right] \\ &\le \mathbb{E}\left[\mathbf{1}\left[V \le K\right] \cdot \frac{\mathbb{E}\left[U^2 \mid \mathcal{F}\right]}{M^2}\right] \\ &\le \mathbb{E}\left[\mathbf{1}\left[V \le K\right] \cdot \frac{V}{M^2}\right] \\ &\le \frac{K}{M^2}. \end{aligned}$$

Choose M large enough that $K/M^2 < \epsilon/2$. Then $Pr\left(|U| > M\right) < \epsilon$ for sufficiently large N, i.e. $U = O_p(1)$.

If $V = o_p(1)$, fix $\epsilon > 0$ and $\delta > 0$. Choose $K > 0$ small enough that $K/\epsilon^2 < \delta/2$. There exists N_0 such that $Pr\left(V > K\right) < \epsilon/2$ for all $N > N_0$. Then by an analogous conditional Markov argument,

$$\begin{aligned} Pr\left(|U| > \varepsilon\right) &\le Pr\left(V > K\right) + Pr\left(|U| > \varepsilon, V \le K\right) \\ &\le \delta/2 + K/\epsilon^2 < \delta \end{aligned}$$

for N large enough, which means $U = o_p(1)$. □

Lemma C2. *Let A be an $N \times N$ matrix measurable with respect to information set $\mathcal{F}$ with $\|A\|_{op} = O_p(1)$; v be an $N \times 1$ vector measurable with respect to $\mathcal{F}$ with $\frac{1}{N}\|v\|^2 = o_p(1)$; and u be an $N \times 1$ vector such that $\mathbb{E}\left[u \mid \mathcal{F}\right] = 0$ and $\|\mathrm{Var}\left[u \mid \mathcal{F}\right]\|_{op} = O_p(1)$. Then*

$$\frac{1}{\sqrt{N}} v' A u = o_p(1).$$

Proof. We have:

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}v'Au\right)^2 \mid \mathcal{F}\right] &= \frac{1}{N}\mathbb{E}\left[v'Auu'A'v \mid \mathcal{F}\right] \\
&= \frac{1}{N}v'A\mathbb{E}\left[uu' \mid \mathcal{F}\right]A'v \\
&\leq \left(\frac{1}{N}\|v\|^2\right)\|A\|_{\text{op}}^2\left\|\mathbb{E}\left[uu' \mid \mathcal{F}\right]\right\|_{\text{op}} \\
&= \left(\frac{1}{N}\|v\|^2\right)\|A\|_{\text{op}}^2\left\|\text{Var}\left[u \mid \mathcal{F}\right]\right\|_{\text{op}} \\
&= o_p(1)O_p(1)O_p(1) = o_p(1),
\end{aligned}$$

with this bound $\mathcal{F}$-measurable. By Lemma C1, $\frac{1}{\sqrt{N}}v'Au = o_p(1)$. □

Lemma C3. *Let v be an $N\times 1$ vector measurable with respect to information set $\mathcal{F}$ with $\|v\|^2 = O_p(1)$ and u be an $N \times 1$ vector such that $\mathbb{E}\left[u \mid \mathcal{F}\right] = 0$ and $\left\|\text{Var}\left[u \mid \mathcal{F}\right]\right\|_{op} = O_p(1)$. Then $v'u = O_p(1)$.*

Proof. We have

$$\begin{aligned}
\mathbb{E}\left[\left(v'u\right)^2 \mid \mathcal{F}\right] &= v'\mathbb{E}\left[uu' \mid \mathcal{F}\right]v \\
&\leq \left\|\mathbb{E}\left[uu' \mid \mathcal{F}\right]\right\|_{\text{op}}\|v\|^2 \\
&= \left\|\text{Var}\left[u \mid \mathcal{F}\right]\right\|_{\text{op}}\|v\|^2 \\
&= O_p(1)O_p(1) = O_p(1),
\end{aligned}$$

with this bound $\mathcal{F}$-measurable. By Lemma C1, $v'u = O_p(1)$. □

Lemma C4. *Suppose $\frac{r_N}{N}\left(\hat{h}-h\right)'\hat{D}^{-1}\varepsilon \xrightarrow{p} 0$, $\frac{r_N}{N}h'\left(\hat{D}^{-1}-D^{-1}\right)\varepsilon \xrightarrow{p} 0$, and $\frac{1}{N}\hat{z}^{*\prime}x \xrightarrow{p} \Pi \neq 0$. Then $r_N\left(\hat{\beta}\left[\hat{z}^*\right]-\beta\right)$ converges to the same distribution as $r_N\left(\hat{\beta}\left[z^*\right]-\beta\right)$.*

Proof. We have

$$r_N\left(\hat{\beta}-\beta\right) = \frac{\frac{r_N}{N}\hat{z}^{*\prime}\varepsilon}{\frac{1}{N}\hat{z}^{*\prime}x}$$

with $\frac{1}{N}\hat{z}^{*\prime}x - \frac{1}{N}z^{*\prime}x \xrightarrow{p} 0$. Moreover,

$$\begin{aligned}
\frac{r_N}{N}\hat{z}^{*\prime}\varepsilon - \frac{r_N}{N}z^{*\prime}\varepsilon &= \frac{r_N}{N}\left(\hat{D}^{-1}\hat{h}-D^{-1}h\right)'\varepsilon \\
&= \frac{r_N}{N}h'\left(\hat{D}^{-1}-D^{-1}\right)\varepsilon + \frac{r_N}{N}\left(\hat{h}-h\right)'\hat{D}^{-1}\varepsilon \xrightarrow{p} 0
\end{aligned}$$

by the assumptions. Thus, given $\frac{1}{N}z^{*\prime}x \xrightarrow{p} \Pi \neq 0$,

$$r_N\left(\hat{\beta}\left[\hat{z}^*\right]-\beta\right) = \frac{\frac{r_N}{N}z^{*\prime}\varepsilon + o_p(1)}{\frac{1}{N}z^{*\prime}x + o_p(1)} = r_N\left(\hat{\beta}\left[z^*\right]-\beta\right) + o_p(1),$$

implying the proposition's claim. $\square$

Proof of Proposition A6. We verify the conditions for Lemma C4. First consider

$$\frac{r_N}{N}\delta'\hat{D}^{-1}\varepsilon = \underbrace{\frac{r_N}{N}\delta'\hat{D}^{-1}\tilde{\varepsilon}}_{(A)} + \underbrace{\frac{r_N}{N}\delta'\hat{D}^{-1}\hat{\psi}}_{(B)} + \underbrace{\frac{r_N}{N}\delta'\hat{D}^{-1}\left(\psi - \hat{\psi}\right)}_{(C)}.$$

Term (A) is $o_p(1)$ by applying Lemma C2 with $\mathcal{F} = (g, w)$, $A = \hat{D}^{-1}$, $u = \tilde{\varepsilon}$, and $v = \frac{r_N}{\sqrt{N}}\delta$. Here $\|\mathbb{E}[\tilde{\varepsilon}\tilde{\varepsilon}' \mid g, w]\|_{\text{op}} = \|\Omega\|_{\text{op}} = O_p(1)$ and $\left\|\hat{D}^{-1}\right\|_{\text{op}} = \lambda_{min}^{-1}\left(\hat{D}\right) = \lambda_{min}^{-1}\left(\hat{\Omega} + \hat{\psi}\hat{\psi}'\right) \leq \lambda_{min}^{-1}\left(\hat{\Omega}\right) \leq c^{-1}$ with probability approaching 1, which implies $\left\|\hat{D}^{-1}\right\|_{\text{op}} = O_p(1)$. Moreover, $\left(\frac{r_N^2}{N}\right) \cdot \frac{1}{N}\|\delta\|^2 = O(1) \cdot \frac{1}{N}\|\delta\|^2 = o_p(1)$.

For term (B) we have:

$$\begin{aligned}
\left|\frac{r_N}{N}\delta'\hat{D}^{-1}\hat{\psi}\right| &\leq \frac{r_N}{N}\left\|\hat{h} - h\right\| \cdot \left\|\hat{D}^{-1}\hat{\psi}\right\| \\
&= \frac{r_N}{\sqrt{N}}\sqrt{\frac{1}{N}\sum_i \left(\hat{h}_i - h_i\right)^2} \cdot \left\|\hat{D}^{-1}\hat{\psi}\right\|.
\end{aligned}$$

Using the Sherman-Morrison formula for $\hat{D} = \hat{\Omega} + \hat{\psi}\hat{\psi}'$,

$$\hat{D}^{-1}\hat{\psi} = \hat{\Omega}^{-1}\left(I - \frac{\hat{\psi}\hat{\psi}'\hat{\Omega}^{-1}}{1 + \hat{\psi}'\hat{\Omega}^{-1}\hat{\psi}}\right)\hat{\psi} = \frac{\hat{\Omega}^{-1}\hat{\psi}}{1 + \hat{\psi}'\hat{\Omega}^{-1}\hat{\psi}}$$

where

$$\begin{aligned}
\left\|\hat{\Omega}^{-1}\hat{\psi}\right\|^2 &= \left(\hat{\psi}'\hat{\Omega}^{-1/2}\right)\hat{\Omega}^{-1}\left(\hat{\Omega}^{-1/2}\hat{\psi}\right) \\
&\leq \left\|\hat{\Omega}^{-1}\right\|_{\text{op}}\hat{\psi}'\hat{\Omega}^{-1}\hat{\psi}
\end{aligned}$$

such that

$$\begin{aligned}
\left\|\hat{D}^{-1}\hat{\psi}\right\| &\leq \left\|\hat{\Omega}^{-1}\right\|_{\text{op}}^{1/2} \cdot \frac{\sqrt{\hat{\psi}'\hat{\Omega}^{-1}\hat{\psi}}}{1 + \hat{\psi}'\hat{\Omega}^{-1}\hat{\psi}} \\
&\leq \frac{1}{2}\left\|\hat{\Omega}^{-1}\right\|_{\text{op}}^{1/2} \leq \frac{1}{2}c^{-1/2}
\end{aligned}$$

with probability approaching 1 since $\sup_{x \geq 0}\frac{\sqrt{x}}{1+x} = \frac{1}{2}$. Therefore $\left\|\hat{D}^{-1}\hat{\psi}\right\| = O_p(1)$ and $\left|\frac{r_N}{N}\delta'\hat{D}^{-1}\hat{\psi}\right| = o_p(1) \cdot O_p(1) = o_p(1)$.

For term (C), we apply Lemma C2 with $\mathcal{F} = w$, $A = \hat{D}^{-1}$, $u = \delta$, and $v = \frac{r_N}{\sqrt{N}}\left(\psi - \hat{\psi}\right)$. Combining terms (A)–(C), we conclude that $\frac{r_N}{N}\delta'\hat{D}^{-1}\varepsilon = o_p(1)$.

Now consider:

$$\frac{r_N}{N}h'\left(\hat{D}^{-1}-D^{-1}\right)\varepsilon = \underbrace{\frac{r_N}{N}h'\left(\hat{D}^{-1}-D^{-1}\right)\tilde{\varepsilon}}_{(D)} + \underbrace{\frac{r_N}{N}h'\left(\hat{D}^{-1}-D^{-1}\right)\psi}_{(E)}.$$

For term (D), by Sherman–Morrison,

$$\frac{r_N}{N}h'\left(\hat{D}^{-1}-D^{-1}\right)\tilde{\varepsilon} = \underbrace{\frac{r_N}{N}h'\left(\hat{\Omega}^{-1}-\Omega^{-1}\right)\tilde{\varepsilon}}_{(D1)} + \underbrace{\frac{r_N}{N}h'\frac{\Omega^{-1}\psi\psi'\Omega^{-1}}{1+\psi'\Omega^{-1}\psi}\tilde{\varepsilon}}_{(D2)} - \underbrace{\frac{r_N}{N}h'\frac{\hat{\Omega}^{-1}\hat{\psi}\hat{\psi}'\hat{\Omega}^{-1}}{1+\hat{\psi}'\hat{\Omega}^{-1}\hat{\psi}}\tilde{\varepsilon}}_{\text{(D3)}}.$$

For (D1) we apply Lemma C2 with $\mathcal{F}=(g,w)$, $A=I$, $u=\tilde{\varepsilon}$, and $v=\frac{r_N}{\sqrt{N}}\left(\hat{\Omega}^{-1}-\Omega^{-1}\right)h$ where

$$\begin{aligned}\frac{1}{\sqrt{N}}\|v\| &= \frac{r_N}{N}\left\|\left(\hat{\Omega}^{-1}-\Omega^{-1}\right)h\right\| \\ &\le \frac{r_N}{\sqrt{N}}\left\|\hat{\Omega}^{-1}-\Omega^{-1}\right\|_{\text{op}}\cdot\frac{1}{\sqrt{N}}\|h\| \\ &= O(1)\cdot o_p(1)\cdot O_p(1) = o_p(1).\end{aligned}$$

Here $\frac{1}{\sqrt{N}}\|h\| = O_p(1)$ because $\mathbb{E}\left[\frac{1}{N}\|h\|^2 \mid w\right] = \frac{1}{N}\operatorname{tr}\mathbb{E}\left[hh' \mid w\right] \le \left\|\mathbb{E}\left[hh' \mid w\right]\right\|_{\text{op}} = \left\|\operatorname{Var}\left[h \mid w\right]\right\|_{\text{op}} = O_p(1)$ and applying Lemma C1.

For (D2), let

$$\zeta = \frac{\Omega^{-1}\psi}{\sqrt{1+\psi'\Omega^{-1}\psi}}$$

so

$$(D2) = \frac{r_N}{N}(h'\zeta)(\zeta'\tilde{\varepsilon}).$$

Note

$$\begin{aligned}\|\zeta\|^2 &= \frac{\psi'\Omega^{-2}\psi}{1+\psi'\Omega^{-1}\psi} \\ &\le \left\|\Omega^{-1}\right\|_{\text{op}}\frac{\psi'\Omega^{-1}\psi}{1+\psi'\Omega^{-1}\psi} \\ &\le \left\|\Omega^{-1}\right\|_{\text{op}} = O_p(1).\end{aligned}$$

Then $h'\zeta = O_p(1)$ by Lemma C3 with $\mathcal{F}=w$, $u=h$, and $v=\zeta$. Moreover $\zeta'\tilde{\varepsilon} = O_p(1)$ by Lemma C3, with $\mathcal{F}=(g,w)$, $u=\tilde{\varepsilon}$, and $v=\zeta$. Hence $(D2) = \frac{r_N}{N}O_p(1) = o_p(1)$ as $r_N/N \to 0$. The same steps show $(D3) = o_p(1)$, by defining

$$\hat{\zeta} = \frac{\hat{\Omega}^{-1}\hat{\psi}}{\sqrt{1 + \hat{\psi}'\hat{\Omega}^{-1}\hat{\psi}}},$$

noting that $\left\|\hat{\zeta}\right\|^2 \leq \left\|\hat{\Omega}^{-1}\right\|_{\text{op}} = O_p(1)$, and following the same steps as with ζ.

For term (E), write

$$\frac{r_N}{N} h' \left(\hat{D}^{-1} - D^{-1}\right) \psi = \underbrace{\frac{r_N}{N} h' \hat{D}^{-1} \hat{\psi}}_{(E1)} + \underbrace{\frac{r_N}{N} h' \hat{D}^{-1} \left(\psi - \hat{\psi}\right)}_{(E2)} - \underbrace{\frac{r_N}{N} h' D^{-1} \psi}_{(E3)}.$$

For (E1), recall from the proof for (B) that $\left\|\hat{D}^{-1}\hat{\psi}\right\| = O_p(1)$. Then $h'\hat{D}^{-1}\hat{\psi} = O_p(1)$ by Lemma C3, with $\mathcal{F} = w$, $u = h$, and $v = \hat{D}^{-1}\hat{\psi}$. Hence $(E1) = \frac{r_N}{N} O_p(1) = o_p(1)$ as $r_N/N \to 0$. For (E3), analogous steps show $\left\|D^{-1}\psi\right\| = O_p(1)$ and so $\frac{r_N}{N} h' D^{-1} \psi = o_p(1)$ again by Lemma C3, with $v = D^{-1}\psi$. Finally, for (E2), we apply Lemma C2 with $\mathcal{F} = w$, $A = \hat{D}^{-1}$, $u = h$, and $v = \frac{r_N}{\sqrt{N}} \left(\psi - \hat{\psi}\right)$. Thus, we conclude that $\frac{r_N}{N} h' \left(\hat{D}^{-1} - D^{-1}\right) \psi = o_p(1)$.

Finally, we turn to the first stage. Using Sherman-Morrison,

$$\begin{aligned}
\frac{1}{N}\hat{z}^{*\prime}x - \frac{1}{N}z^{*\prime}x &= \frac{1}{N}\hat{h}'\hat{D}^{-1}x - \frac{1}{N}h'D^{-1}x \\
&= \frac{1}{N}\left(\hat{h} - h\right)'\hat{D}^{-1}x + \frac{1}{N}h'\left(\hat{D}^{-1} - D^{-1}\right)x \\
&= \frac{1}{N}\left(\hat{h} - h\right)'\hat{D}^{-1}x + \frac{1}{N}h'\left(\hat{\Omega}^{-1} - \Omega^{-1} + \frac{\Omega^{-1}\psi \cdot \psi'\Omega^{-1}}{1 + \psi'\Omega^{-1}\psi} - \frac{\hat{\Omega}^{-1}\hat{\psi} \cdot \hat{\psi}'\hat{\Omega}^{-1}}{1 + \hat{\psi}'\hat{\Omega}^{-1}\hat{\psi}}\right)x \\
&= \underbrace{\frac{1}{N}\delta'\hat{D}^{-1}x}_{(F)} + \underbrace{\frac{1}{N}h'\left(\hat{\Omega}^{-1} - \Omega^{-1}\right)x}_{(G)} + \underbrace{\frac{1}{N}h'\zeta \cdot \zeta'x}_{(H)} - \underbrace{\frac{1}{N}h'\hat{\zeta} \cdot \hat{\zeta}'x}_{(I)}.
\end{aligned}$$

Here

$$\begin{aligned}
|(F)| &\leq \left(\frac{1}{N}\|\delta\|^2\right)^{1/2} \left\|\hat{D}^{-1}\right\|_{\text{op}} \left(\frac{1}{N}\|x\|^2\right)^{1/2} = o_p(1)O_p(1)O_p(1) = o_p(1), \\
|(G)| &\leq \left(\frac{1}{N}\|h\|^2\right)^{1/2} \left\|\hat{\Omega}^{-1} - \Omega^{-1}\right\|_{\text{op}} \left(\frac{1}{N}\|x\|^2\right)^{1/2} = O_p(1)o_p(1)O_p(1) = o_p(1),
\end{aligned}$$

For term (H), we have established earlier that $h'\zeta = O_p(1)$. Moreover, $\frac{1}{\sqrt{N}}|\zeta'x| \leq \|\zeta\| \cdot \frac{1}{\sqrt{N}}\|x\| = O_p(1)$ such that $\frac{1}{N}h'\zeta \cdot \zeta'x = O_p(1/\sqrt{N}) = o_p(1)$. The same argument applies to term (I).

With all conditions of Lemma C4 verified, we conclude that $\hat{\beta}\left[\hat{z}^*\right]$ has the same asymptotic distribution as $\hat{\beta}\left[z^*\right]$.

Online Appendix for *Optimal Formula Instruments*

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D Additional Results

D.1 Heterogeneous Treatment Effects

This appendix shows that recentered IV estimators can identify a convex average of heterogeneous treatment effects under natural extensions of the classic assumptions in Imbens and Angrist (1994). Consider a general causal model with potential outcome functions $y_i(\cdot)$ and observed outcomes given by $y_i = y_i(x_i)$. We write $y(\cdot)$ for the vector collecting $y_i(\cdot)$. For notational simplicity we suppose x_i is continuously distributed and bounded from below and that $y_i(\cdot)$ is differentiable; treatment effects are then given as $\beta_i(x) = \frac{\partial}{\partial x} y_i(x)$ for treatment margin x. We further normalize the lower bound of x_i to zero without loss. All of these assumptions are straightforward to relax or modify under appropriate regularity conditions (see, e.g., Kolesár and Plagborg-Møller (2025)).

The potential outcome notation implicitly makes an exclusion restriction: that any effects of g on y_i are only through $x_i = h_i(g, w, u)$. We further assume a stronger form of shock exogeneity:

Assumption D1. *(Independence):* $g \perp\!\!\!\perp y(\cdot) \mid w$.

This assumption is stronger than the mean-independence Assumption 1 since conditions like Assumption 1 are generally not sufficient to ensure convex weighting of heterogeneous effects (Borusyak and Hull 2024). Assumption D1 is nevertheless standard (e.g., it follows Imbens and Angrist (1994)) and holds by design when the shocks arise from a randomized controlled trial.

Finally, for a recentered instrument $z \in \mathfrak{R}$, we consider a stochastic monotonicity assumption:

Assumption D2. *(Monotonicity):* $\text{Cov}\left[\mathbf{1}\left[x_i \geq x\right], z_i \mid y(\cdot), w\right] \geq 0$ *almost-surely for each* x *and* i.

We then have the following result for the IV estimand $\beta\left[z\right] = \frac{\mathbb{E}\left[\sum_i z_i y_i\right]}{\mathbb{E}\left[\sum_i z_i x_i\right]}$.

Proposition D1. *Under Assumptions D1 and D2,*

$$\beta\left[z\right] = \mathbb{E}\left[\frac{1}{N}\sum_i \int_0^\infty \beta_i(x)\omega_i(x)dx\right]$$

where

$$\omega_i(x) = \frac{\phi_i(x)\sigma_i^2}{\mathbb{E}\left[\frac{1}{N}\sum_i \int_0^\infty \phi_i(x')\sigma_i^2 dx'\right]} \geq 0 \ \ a.s.$$

gives a convex weighting scheme, for $\phi_i(x) = \frac{\text{Cov}[\mathbf{1}[x_i \geq x], z_i \mid y(\cdot), w]}{\text{Var}[z_i \mid y(\cdot), w]}$ *and* $\sigma_i^2 = \text{Var}\left[z_i \mid w\right]$.

Proof. Write $y_i = y_i(0) + \int_0^{x_i} \beta_i(x)dx$ and note that $\mathbb{E}[z_i y_i(0)] = \mathbb{E}[\mathbb{E}[z_i \mid y(0), w]\, y_i(0)] = \mathbb{E}[\mathbb{E}[z_i \mid w]\, y_i(0)] = 0$ by iterated expectations, Assumption D1, and $z \in \mathfrak{R}$. Thus,

$$\begin{aligned}
\mathbb{E}[z_i y_i] &= \mathbb{E}\left[z_i \int_0^{x_i} \beta_i(x)dx\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_0^{x_i} \beta_i(x) z_i dx \mid y(\cdot), w\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_0^{\infty} \beta_i(x) z_i \mathbf{1}[x_i \geq x]\, dx \mid y(\cdot), w\right]\right] \\
&= \mathbb{E}\left[\int_0^{\infty} \beta_i(x) \mathbb{E}[z_i \mathbf{1}[x_i \geq x] \mid y(\cdot), w]\, dx\right] \\
&= \mathbb{E}\left[\int_0^{\infty} \beta_i(x) \phi_i(x) \sigma_i^2 dx\right]
\end{aligned}$$

where the final line uses $\mathbb{E}[z_i \mid y(\cdot), w] = 0$ and $\text{Var}[z_i \mid y(\cdot), w] = \text{Var}[z_i \mid w_i] = \sigma_i^2$ to show that $\mathbb{E}[z_i \mathbf{1}[x_i \geq x] \mid y(\cdot), w] = \text{Cov}[z_i, \mathbf{1}[x_i \geq x] \mid y(\cdot), w] = \phi_i(x)\sigma_i^2$. Similar steps show

$$\mathbb{E}[z_i x_i] = \mathbb{E}\left[\int_0^{\infty} \phi_i(x) \sigma_i^2 dx\right],$$

completing the proof. □

The result shows that different recentered IV estimands $\beta[z]$ generally identify different weighted averages of treatment effects $\beta_i(x)$. The weights are driven by two factors: the slope coefficient $\phi_i(x)$ in the conditional-on-$(y(\cdot), w)$ observation-specific population regression of $\mathbf{1}[x_i \geq x]$ on z_i, and the conditional-on-w variance of the recentered IV, σ_i^2. Assumption D2 makes $\omega_i(x)$ non-negative.

The monotonicity assumption here is similar to those in Small et al. (2017) and Borusyak and Hull (2024) by restricting the conditional relationship between the treatment and the recentered instrument given potential outcomes and w, rather than defining potential treatment functions as in Imbens and Angrist (1994).[40] If such $x_i(z)$ are well-defined and weakly increasing in z then Assumption D2 holds. More generally Assumption D2 can hold when the relationship between $x_i = h_i(g, w, u)$ and z_i is not causal, such as when $u \neq \emptyset$ and $z_i = p_i(g, w) - \mathbb{E}[p_i(g, w) \mid w]$ for some function $p_i(\cdot)$. See Appendix C of Borusyak and Hull (2024) for an example in the shift-share case.

In the special case where the treatment $x_i = h_i(g, w)$ is a function of g and w only and $z_i = x_i - \mathbb{E}[h_i(g, w) \mid w]$, Assumption D2 holds automatically: $\text{Cov}[x_i, \mathbf{1}[x_i \geq x] \mid y(\cdot), w] \geq 0$ implies $\phi_i(x) \geq 0$. If furthermore treatment effects are linear, $y_i = \beta_i x_i + \varepsilon_i$, then Proposition D1 shows $\beta[z]$ identifies a conditional-variance-weighted average of the β_i, $\mathbb{E}\left[\frac{1}{N}\sum_i \beta_i \sigma_i^2\right] / \mathbb{E}\left[\frac{1}{N}\sum_i \sigma_i^2\right]$, extending a classic result in Angrist (1998). If z_i is not only recentered by $\mathbb{E}[x_i \mid w]$ but also rescaled by $\text{Var}[x_i \mid w]^{-1}$, Proposition D1 shows that $\beta[z]$ identifies an unweighted average treatment effect,

[40] Small et al. (2017) consider stochastic monotonicity only for binary treatments while Assumption 3 in Borusyak and Hull (2024) is stronger than the above condition by restricting the conditional distribution of the treatment given the instrument. By restricting only the conditional covariance of $\mathbf{1}[x_i \geq x]$ and z_i, Assumption D2 better accommodates formula instruments; see Appendix C of Borusyak and Hull (2024) for a discussion.

$\mathbb{E}\left[\frac{1}{N}\sum_i \beta_i\right]$.

Proposition D1 also implies that dropping observations with $\operatorname{Var}[z_i \mid w] = 0$ does not change the estimand, as $\phi_i(x) = 0$ for them. Thus, while our application focused on the sample of "exposed" individuals for efficiency reasons assuming constant effects, this choice has no consequences for the interpretation of the estimates even when the effects are heterogeneous.

D.2 Including Fixed Effects

This appendix shows how the framework can be extended to settings where a set of fixed effects, like the state effects in the Section 4 application, are necessary in estimation. We consider a repeated cross section of individuals i in years $t(i)$ and states $s(i)$ and a set of state-level shocks $g = \{g_s\}$ to match our application; the ideas apply more generally.[41]

Consider the following causal model (illustrated with a diagram in Appendix Figure A1):

$$y_{it} = \beta x_{it} + \alpha_{s(i)} + \varepsilon_{it},$$

where here, in the repeated cross section, $t = t(i)$ and, as before, $x_{it} = h_{it}(g, w, u)$. The $\alpha_{s(i)}$ capture unobserved time-invariant factors that affect the outcome additively. We assume there are two groups of such factors: $\alpha_s = \alpha(\gamma_s, \delta_s)$. The former, γ_s, is a confounder that affects expansion decisions g_s but is statistically independent of the characteristics of individuals in the states, $w = \{c_{it}, s(i)\}$. The latter, δ_s, is statistically independent of γ_s and g but potentially related to w and u. For example, γ_s might capture idiosyncratic preferences of the state's governor while δ_s captures long-run economic conditions in the state.[42] The error term $\varepsilon_{it(i)}$ can be arbitrarily related to w.

Shocks are exogenous in this model, in the sense of Assumption 1, only after removing state fixed effects: $\mathbb{E}\left[\varepsilon_{it(i)} \mid g, w\right] = \mathbb{E}\left[\varepsilon_{it(i)} \mid w\right]$, but generally $\mathbb{E}\left[\alpha_{s(i)} + \varepsilon_{it(i)} \mid g, w\right] \neq \mathbb{E}\left[\alpha_{s(i)} + \varepsilon_{it(i)} \mid w\right]$. Furthermore, a candidate formula instrument $z_{it} = f_{it}(g, w)$ can be recentered by using permutations of g as counterfactual shocks. This is because the dependence of g_s and γ_s does not affect the distribution of g given w and thus permutations of g are equally likely conditional on w. As usual, recentering is generally needed here since $\mathbb{E}\left[\varepsilon_{it} \mid w\right]$ is unrestricted.

Our theoretical results for optimal formula instruments apply here on the residualized model

$$\tilde{y}_{it} = \beta \tilde{x}_{it} + \tilde{\varepsilon}_{it}, \tag{A2}$$

where $\tilde{v}_{it}$ denotes the residual from a sample projection of generic variable v_{it} on state-of-residence indicators (see Appendix A.1). Indeed, since $s(i)$ is included in w, such residualization involves premultiplying the vectors of y_{it} and x_{it} (and therefore of ε_{it}) by an annihilator matrix $M(w)$.

[41] Relative to the application, we simplify some details—such as considering simple permutations of expansion shocks that do not condition on the state governor's party affiliation.

[42] Appendix Figure A1 indicates that δ_s affects the demographics in c_{it} and other policies in u and those variables affect of y_{it}, but the nature and direction of these relationships is immaterial. Note also that we draw the arrow from γ_s to g_s and not the other way around only because it is natural to think that the expansion shocks arise later than the time-invariant factors in γ_s.

Thus Assumption 1 holds for $\tilde{\varepsilon}_{it}$, since $\mathbb{E}[\tilde{\varepsilon} \mid g, w] = \mathbb{E}[M(w)\varepsilon \mid g, w] = \mathbb{E}[M(w)\varepsilon \mid w] = \mathbb{E}[\tilde{\varepsilon} \mid w]$. Moreover, if $\mathbb{E}[\varepsilon\varepsilon' \mid g, w] = \mathbb{E}[\varepsilon\varepsilon' \mid w]$ as in Theorem 1, we similarly have $\mathbb{E}[\tilde{\varepsilon}\tilde{\varepsilon}' \mid g, w] = \mathbb{E}[\tilde{\varepsilon}\tilde{\varepsilon}' \mid w]$.

Our algorithm for approximating the optimal instrument also applies. The key step is to approximate $\mathbb{E}[\tilde{x} \mid g, w] - \mathbb{E}[\tilde{x} \mid w] = M(w)\left(\mathbb{E}[x \mid g, w] - \mathbb{E}[x \mid w]\right)$ (with $\tilde{x}$ and x collecting $\tilde{x}_{it}$ and x_{it}, respectively), which can be accomplished by forming a prediction $p_{it}(g, w)$ of x_{it}, recentering it by $\mathbb{E}[p_{it}(g, w) \mid w]$ (e.g., by permuting state shocks), and residualizing the recentered prediction by state indicators. The last step can be automated by including state indicators in estimation, as we do in the application.

D.3 Reconciling Simulated and Recentered IV Estimates

This appendix first discusses how measurement error in Medicaid eligibility calculations can bias both simulated and recentered IV estimates of eligibility effects. We then show how such bias is avoided in IV regressions which use Medicaid enrollment as the endogenous variable instead.

For clarity of the theoretical discussion, we simplify the setup. First, we suppose that a single *iid* cross-section of 2014 data is available and state fixed effects are not included; we correspondingly drop the t subscript throughout. Second, we assume that state decisions to expand Medicaid coverage are unconditionally as-good-as-random and mutually independent with the same probability. We then consider the simple causal model of $y_i = \beta x_i^* + \varepsilon_i$ where y_i is a measure of insurance coverage for individual i and $x_i^* \in \{0, 1\}$ is i's true Medicaid eligibility. We suppose changes in x_i^* come from the exogenous expansion of eligibility policy; formally, we write $x_i^* = x_{i0}^* + e_i^* g_i$ where $x_{i0}^*, e_i^*, g_i \in \{0, 1\}$ with $x_{i0}^* e_i^* = 0$. Here the x_{i0}^* indicator switches on for individuals who would be eligible for Medicaid regardless of the binary expansion shock g_i, while e_i^* is on for those who become eligible when $g_i = 1$ (i.e. exposed individuals). Individuals who are never eligible regardless of g_i correspond to $x_{i0}^* = e_i^* = 0$. Rather than observing x_{i0}^* and e_i^* directly, we assume the researcher computes eligibility x_i from mismeasured x_{i0} and e_i: i.e., $x_i = x_{i0} + e_i g_i$ where again $x_{i0}, e_i \in \{0, 1\}$ and $x_{i0} e_i = 0$. Such measurement error could reflect error in self-reported household income or demographics (Brooks 2019). We assume the as-good-as-random expansion shocks only affect outcomes by changing eligibility, making them exogenous: $g_i \perp\!\!\!\perp (\varepsilon_i, x_{i0}^*, e_i^*, x_{i0}, e_i)$.

We first show how simulated and recentered IV estimates of β, which use measured eligibility as the right-hand side variable, are biased by this measurement error. The simulated IV estimate uses the expansion shock to instrument measured eligibility in the full sample of individuals. This IV regression identifies

$$\begin{aligned}
\beta^{SI} &= \frac{\text{Cov}[y_i, g_i]}{\text{Cov}[x_i, g_i]} \\
&= \frac{\text{Cov}[\beta(x_{i0}^* + e_i^* g_i) + \varepsilon_i, g_i]}{\text{Cov}[x_{i0} + e_i g_i, g_i]} \\
&= \beta \frac{\mathbb{E}[e_i^*]}{\mathbb{E}[e_i]},
\end{aligned} \tag{A3}$$

where we use the independence of g_i in the third line. The recentered IV estimate of β, implemented as in Panel A of Table 2, regresses the outcome on measured eligibility in the sample of individuals who are exposed according to the observed e_i. This estimator identifies

$$\begin{aligned}\tilde{\beta} &= \frac{\mathrm{Cov}\left[y_i, x_i \mid e_i = 1\right]}{\mathrm{Var}\left[x_i \mid e_i = 1\right]} \\ &= \frac{\mathrm{Cov}\left[\beta(x_{i0}^* + e_i^* g_i) + \varepsilon_i, g_i \mid e_i = 1\right]}{\mathrm{Var}\left[g_i \mid e_i = 1\right]} \\ &= \beta\mathbb{E}\left[e_i^* \mid e_i = 1\right], \end{aligned} \tag{A4}$$

where we again use the independence of g_i in the third line. Since $\mathbb{E}\left[e_i^* \mid e_i = 1\right] \in [0, 1)$ whenever $Pr(e_i^* = 0, e_i = 1) > 0$ and $\mathbb{E}\left[e_i^* \mid e_i = 1\right] = \mathbb{E}\left[e_i^* e_i\right] / \mathbb{E}\left[e_i\right] < \mathbb{E}\left[e_i^*\right] / \mathbb{E}\left[e_i\right]$ whenever $Pr(e_i^* = 1, e_i = 0) > 0$, these expressions show that the recentered IV estimand is generally attenuated relative to both the causal parameter of interest and the simulated IV estimand: $\left|\tilde{\beta}\right| < |\beta|$ and $\left|\tilde{\beta}\right| < \left|\beta^{SI}\right|$. The simulated IV estimand can either be larger or smaller than the causal parameter, depending on the relative shares of true and computed exposure $\mathbb{E}\left[e_i^*\right]$ and $\mathbb{E}\left[e_i\right]$.

Such bias, however, does not arise when estimating the effects of Medicaid enrollment using either of the two instruments. This follows because in such IV regressions both the first stage regression (of enrollment on the eligibility instrument) and the reduced form (some outcome, such as private insurance coverage, on the same instrument) have the same proportionate bias given by equations (A3) and (A4), depending on the instrument.

D.4 Efficiency of Full-Sample Recentered IV Estimates

Appendix Table A4 reports recentered IV estimates of Medicaid takeup and crowdout effects in the full sample of 2014 and 2013 individuals, not restricting to the exposed sample as in our baseline specification. Panel B, which includes demographic controls, again finds much narrower confidence intervals relative to the simulated eligibility instrument. However, excluding these controls in Panel A yields an intriguing pattern: confidence intervals for the recentered IV are much wider than those of the simulated instrument.

Here we explain how a combination of two factors generates the discrepancy between panels A and B of the table. First, the regression residuals are strongly correlated with the indicator for an individual being exposed to the ACA expansion experiment, which is not controlled for in this regression. Second, exogenous shocks are assigned at the level of states, which include both exposed and non-exposed individuals. This discussion reveals why the problem does not arise when focusing on the exposed sample or when appropriate controls are included. We further relate this problem to the third step of Algorithm 1.

For clarity of the theoretical discussion, we simplify the setup. First, we suppose that a single 2014 cross-section is available and thus state fixed effects are not included; we correspondingly drop the t subscript throughout. We allow for other controls to be included. Second, we assume states only change eligibility as prescribed by their expansion decision, i.e., $e_k^{\Delta} = \emptyset$. Finally, we assume that state decisions to expand are independent with a known propensity $\mathbb{E}\left[g_k \mid w\right]$ (e.g., as a function of the state governor's party). Thus, the recentered expansion indicator $\tilde{g}_k = g_k - \mathbb{E}[g_k \mid w]$ can be computed without permutations.[43] We compare the recentered simulated instrument z_i^{SI} to the recentered best predictor instrument $\tilde{z}_i$, ignoring controls that span $\mathbb{E}[g_k \mid w]$ (e.g., the state party indicator).

Under these additional assumptions, using the recentered simulated instrument is equivalent to using the recentered expansion indicator: $z_i^{SI} = \tilde{g}_{s(i)}$. The recentered best predictor instrument only differs by setting z_i^{SI} to zero for the non-exposed sample: $\tilde{z}_i = z_i - \mathbb{E}[z_i \mid w] = f_i \tilde{g}_{s(i)}$, where f_i is an indicator for individual i being in the exposed group. With $x_i = z_i$, the first stage can be written $x_i = \mu_i + f_i \tilde{g}_{s(i)}$, where the expected instrument μ_i equals 0 for individuals who are not eligible regardless of $g_{s(i)}$, 1 for those always eligible, and $\mathbb{E}\left[g_{s(i)} \mid w\right]$ otherwise.

We now consider the approximate variances of the two estimators, $\operatorname{Var}\left[\frac{1}{N}\sum_i z_i^{SI}\varepsilon_i^{\perp}\right] / \mathbb{E}\left[\frac{1}{N}\sum_i z_i^{SI} x_i^{\perp}\right]^2$ and $\operatorname{Var}\left[\frac{1}{N}\sum_i \tilde{z}_i \varepsilon_i^{\perp}\right] / \mathbb{E}\left[\frac{1}{N}\sum_i \tilde{z}_i x_i^{\perp}\right]^2$, where $\perp$ denotes the in-sample projection residual on the control variables (including a constant). We focus our attention on the numerators of these expressions because the first-stage covariances in the denominator are asymptotically equivalent (and equal in finite samples without controls).[44] For simplicity of exposition we also consider an individual's state of residence $s(i)$ as fixed. Letting $N_k = \sum_i \mathbf{1}[s(i) = k]$ denote the (fixed) number of individuals in each state k, it can then be shown that

$$\frac{\operatorname{Var}\left[\frac{1}{N}\sum_i \tilde{z}_i \varepsilon_i^{\perp}\right]}{\operatorname{Var}\left[\frac{1}{N}\sum_i z_i^{SI} \varepsilon_i^{\perp}\right]} = \frac{\sum_k \left(\frac{N_k}{N}\right)^2 \operatorname{Var}[\tilde{g}_k]\, \mathbb{E}\left[e_{RI,k}^2\right]}{\sum_k \left(\frac{N_k}{N}\right)^2 \operatorname{Var}[\tilde{g}_k]\, \mathbb{E}\left[e_{SI,k}^2\right]}, \tag{A5}$$

where $e_{RI,k} = \frac{1}{N_k}\sum_{i:\, s(i)=k} \varepsilon_i^{\perp} f_i$ is the sum of residuals of *exposed* individuals in state k (normalized by N_k), while $e_{SI,k} = \frac{1}{N_k}\sum_{i:\, s(i)=k} \varepsilon_i^{\perp}$ averages over *all* observations in the state.[45]

Equation (A5) shows that the recentered IV delivers power gains relative to the simulated instrument approach whenever the normalized sum of residuals is closer to zero for a typical state, in the mean-squared sense, when restricting to exposed individuals. The restricted sum has fewer summands, working in favor of the recentered IV. If the expansion shocks were assigned at the individual level, without state clustering, this would guarantee that the recentered IV is more efficient (since $e_{RI,k} = e_{SI,k}$ for exposed individuals in that case).

However, this simplified example shows that the recentered IV is likely to deliver a power loss when the shocks g_k are clustered and $\varepsilon_i^{\perp}$ is strongly correlated with the indicator of exposed sample f_i (i.e., exposed individuals have systematically different residuals, and f_i is not controlled for). To see this simply, suppose $\mathbb{E}\left[\varepsilon_i^{\perp} \mid f_i = 1, w\right] = \alpha \neq 0$ for all i. In this scenario $e_{RI,k}$ is not mean-zero,

[43] Formally, we assume that w does not include the permutation class of g. Under this assumption, $\tilde{g}_k$ is independent across states conditionally on w, simplifying the analysis.

[44] Namely, since f_i is binary, $\mathbb{E}\left[\frac{1}{N}\sum_i z_i^{SI} x_i\right] = \mathbb{E}\left[\frac{1}{N}\sum_i \tilde{g}_{s(i)}\left(\mu_i + f_i \tilde{g}_{s(i)}\right)\right] = \mathbb{E}\left[\frac{1}{N}\sum_i f_i \tilde{g}_{s(i)}^2\right] = \mathbb{E}\left[\frac{1}{N}\sum_i f_i \tilde{g}_{s(i)}\left(\mu_i + f_i \tilde{g}_{s(i)}\right)\right] = \mathbb{E}\left[\frac{1}{N}\sum_i \tilde{z}_i x_i\right]$. With controls this equality holds asymptotically, since the difference between x_i and $x_i^{\perp}$ is uncorrelated with $z_i^{SI} - \tilde{z}_i = (1 - f_i)\,\tilde{g}_{s(i)}$.

[45] Namely, $\operatorname{Var}\left[\frac{1}{N}\sum_i \tilde{z}_i \varepsilon_i^{\perp}\right] = \sum_k \left(\frac{N_k}{N}\right)^2 \mathbb{E}\left[\left(\frac{1}{N_k}\sum_{i:\, s(i)=k} \tilde{z}_i \varepsilon_i^{\perp}\right)^2\right] = \sum_k \left(\frac{N_k}{N}\right)^2 \mathbb{E}\left[\tilde{g}_k^2 \cdot \left(\frac{1}{N_k}\sum_{i:\, s(i)=k} f_i \varepsilon_i^{\perp}\right)^2\right] = \sum_k \left(\frac{N_k}{N}\right)^2 \operatorname{Var}[\tilde{g}_k]\, \mathbb{E}\left[e_{RI,k}^2\right]$, since $\mathbb{E}\left[\frac{1}{N}\sum_i \tilde{z}_i \varepsilon_i^{\perp}\right] = 0$, and similarly for $\operatorname{Var}\left[\frac{1}{N}\sum_i z_i^{SI} \varepsilon_i^{\perp}\right]$.

even on average across states, which potentially yields a high mean-squared residual:

$$\mathbb{E}\left[e_{RI,k}\right] = \mathbb{E}\left[\mathbb{E}\left[e_{RI,k} \mid w\right]\right] = \mathbb{E}\left[\frac{1}{N_k}\sum_{i:\, s(i)=k}\mathbb{E}\left[\varepsilon_i^{\perp} f_i \mid w\right]\right]$$
$$= \mathbb{E}\left[\frac{1}{N_k}\sum_{i:\, s(i)=k}\mathbb{E}\left[\varepsilon_i^{\perp} \mid f_i = 1, w\right] f_i\right] = \alpha \cdot \mathbb{E}\left[\frac{\sum_{i:\, s(i)=k} f_i}{N_k}\right] \neq 0.$$

The simulated instrument, which does not condition on $f_i = 1$, does not suffer from this problem since $\varepsilon_i^{\perp}$ is mean-zero in the sample. Another interpretation of this problem is that in this case the sums of residuals over the exposed and non-exposed individuals of a given state will tend to have opposite signs, increasing efficiency of the simulated instrument that uses both subsamples.

The predictions of this discussion are borne out in the data. In Panel C of Appendix Table A4 we verify that the confidence interval of recentered IV becomes dramatically narrowed with a single control of f_i (interacted with the 2014 indicator appropriately for the difference-in-differences setting).[46] Moreover, demographic controls in Panel B of Appendix Table A4 capture most of the variation in f_i, delivering similar results. Our recentered IV specifications in the main text, by restricting the sample to the exposed individuals, effectively control for year interacted with f_i.

We note that here controlling for the exposed sample indicator is closely related to our third step in constructing the optimal recentered IV, discussed in Section 3.3: this control happens to play the role of the predetermined predictors of the residual, ψ. Our application therefore highlights that in general there is no guarantee of an efficiency gain from improving the first stage with a recentered IV (i.e., approximating the recentered best predictor) if adjustment for ψ is not feasible.

References

Angrist, Joshua. 1998. "Estimating the labor market impact of voluntary military service using social security data on military applicants." *Econometrica* 66:249–288.

Borusyak, Kirill, and Peter Hull. 2024. "Negative weights are no concern in design-based specifications." In *AEA Papers and Proceedings,* 114:597–600. American Economic Association 2014 Broadway, Suite 305, Nashville, TN 37203.

Brooks, Tricia. 2019. "What Some Researchers Get Wrong About Medicaid's Income Eligibility Requirements." *Health Affairs Blog.*

Imbens, Guido W., and Joshua D. Angrist. 1994. "Identification and Estimation of Local Average Treatment Effects." *Econometrica* 62:467–475.

[46] The efficiency of IV specifications that only control for the expected eligibility prediction in columns 2, 4, and 6 of Panel A is lower because this control does not span f_i.

Kolesár, Michal, and Mikkel Plagborg-Møller. 2025. "Dynamic causal effects in a nonlinear world: the good, the bad, and the ugly." *Journal of Business & Economic Statistics* 43:737–754.

Small, Dylan S, Zhiqiang Tan, Roland R Ramsahai, Scott A Lorch, and M Alan Brookhart. 2017. "Instrumental variable estimation with a stochastic monotonicity assumption."

Online Appendix Figures and Tables

Figure A1: Causal Diagram with Fixed Effects

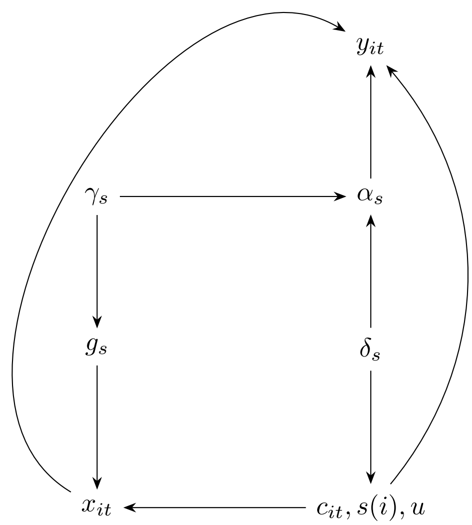
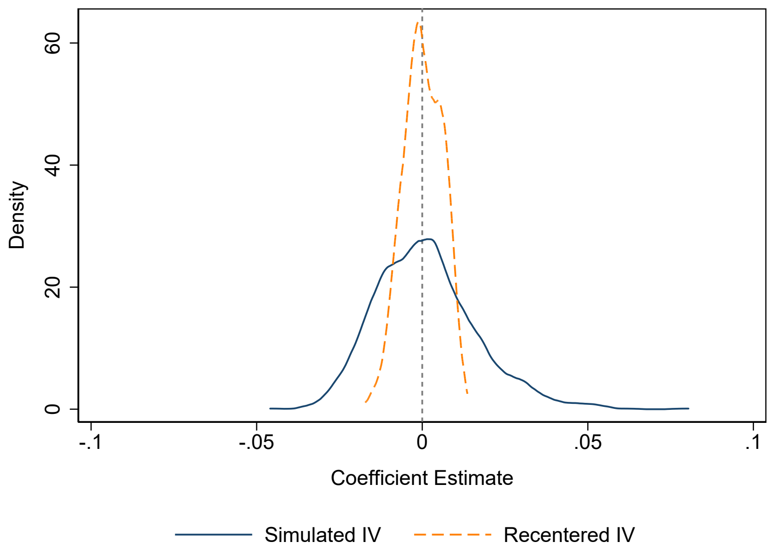
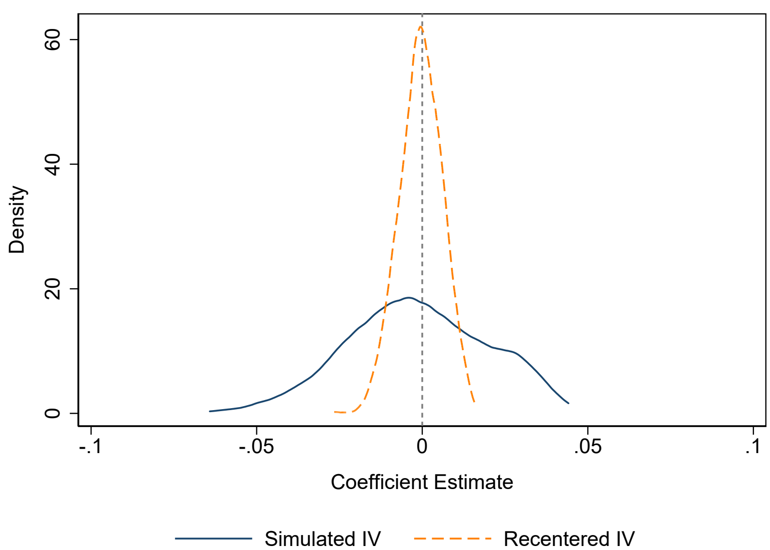


Figure A2: Medicaid Application: Simulated Estimator Distributions

A. Has Medicaid



B. Has Employer-Sponsored Insurance



Notes: This figure plots the simulated distributions of the two IV regression discussed in the text: one using a conventional simulated instrument and the other using as an instrument a recentered prediction of Medicaid eligibility. See Appendix D.4 for a description of the simulation procedure. The true effect of zero in both panels is indicated by the dashed vertical line.

Table A1: Medicaid Application: Second Stage-Estimates with Additional Controls

	Has Medicaid		Has Private Insurance		Has Employer-Sponsored Insurance	
	Simulated IV (1)	Recentered IV (2)	Simulated IV (3)	Recentered IV (4)	Simulated IV (5)	Recentered IV (6)
Panel A. Medicaid Eligibility Effects						
Eligible	0.135	0.073	-0.050	-0.024	0.003	-0.008
	(0.029)	(0.010)	(0.022)	(0.007)	(0.013)	(0.005)
	[0.082,0.224]	[0.050,0.095]	[-0.112,0.001]	[-0.042,-0.008]	[-0.039,0.037]	[-0.019,0.005]
Panel B. Medicaid Enrollment Effects						
Has Medicaid			-0.372	-0.334	0.023	-0.108
			(0.146)	(0.091)	(0.097)	(0.060)
			[-0.767,0.008]	[-0.566,-0.121]	[-0.238,0.322]	[-0.235,0.081]
Exposed Sample	N	Y	N	Y	N	Y
States	43	43	43	43	43	43
Individuals	2,397,313	421,042	2,397,313	421,042	2,397,313	421,042

Notes: Panel A of this table reports second-stage coefficients from the two IV regressions described in the text: one using a conventional simulated instrument and the other using as an instrument a recentered prediction of Medicaid eligibility. Columns 1, 3, and 5 estimate regressions in the full sample of individuals in 2013–2014, while Columns 2, 4, and 6 restrict to the sample of individuals whose characteristics and state of residence make them exposed to the partial ACA Medicaid expansion in 2014. All regressions control for state and year fixed effects, an indicator for Republican-governed states interacted with year, and full interactions of deciles of household income, parental status, work status, and year. Panel B shows estimates from IV regressions which use an indicator for Medicaid enrollment as the endogenous variable, instead of an indicator for Medicaid eligibility. State-clustered standard errors are reported in parentheses; 95% confidence intervals, obtained by a wild score bootstrap, are reported in brackets.

Table A2: Medicaid Application: Pre-Trend Estimates

	Has Medicaid		Has Private Insurance		Has Employer-Sponsored Insurance	
	Simulated IV (1)	Recentered IV (2)	Simulated IV (3)	Recentered IV (4)	Simulated IV (5)	Recentered IV (6)
Panel A. Baseline Specification						
Eligibile	0.022	0.020	−0.015	−0.011	−0.011	−0.007
	(0.009)	(0.004)	(0.017)	(0.004)	(0.017)	(0.005)
	[-0.004,0.042]	[0.009,0.029]	[-0.070,0.023]	[-0.020,-0.003]	[-0.060,0.026]	[-0.019,0.004]
Panel B. With Additional Controls						
Eligibile	0.023	0.020	−0.019	−0.014	−0.016	−0.011
	(0.010)	(0.004)	(0.014)	(0.004)	(0.016)	(0.005)
	[-0.012,0.040]	[0.009,0.027]	[-0.056,0.020]	[-0.022,-0.006]	[-0.051,0.031]	[-0.023,0.001]
Exposed Sample	N	Y	N	Y	N	Y
States	43	43	43	43	43	43
Individuals	2,400,142	425,112	2,400,142	425,112	2,400,142	425,112

Notes: This table reports pre-trend estimates for the two IV regressions described in the text: one using a conventional simulated instrument and the other using as an instrument a recentered prediction of Medicaid eligibility. Pre-trend estimates come from the IV specifications described in the text replacing 2013 individuals with 2012 individuals and replacing 2014 individuals with 2013 individuals. Columns 1, 3, and 5 estimate regressions in the full sample of individuals, while Columns 2, 4, and 6 restrict to the sample of individuals whose characteristics and state of residence make them exposed to the partial ACA Medicaid expansion in 2014. All regressions control for state and year fixed effects and an indicator for Republican-governed states interacted with year. The regressions in Panel B additionally control for deciles of household income, interacted with indicators for parental and work status and year. Conventional state-clustered standard errors are reported in parentheses; 95% confidence intervals, obtained by a wild score bootstrap, are reported in brackets.

Table A3: Medicaid Application: Alternative Designs

	Has Medicaid	Has Private Insurance	Has Employer-Sponsored Insurance
	(1)	(2)	(3)
Panel A. Republican Governor and 2012 Median Income			
Eligible	0.078	−0.016	−0.003
	(0.011)	(0.008)	(0.006)
	[0.051,0.098]	[-0.044,-0.004]	[-0.021,0.005]
Has Medicaid		−0.210	−0.043
		(0.106)	(0.073)
		[-0.592,-0.056]	[-0.242,0.079]
Panel B. Republican Governor, 2012 Median Income, 2012 Medicaid Coverage			
Eligible	0.076	−0.023	−0.009
	(0.012)	(0.007)	(0.005)
	[0.054,0.103]	[-0.040,-0.006]	[-0.021,0.003]
Has Medicaid		−0.302	−0.123
		(0.095)	(0.059)
		[-0.540,-0.081]	[-0.251,0.053]
Exposed Sample	Y	Y	Y
States	43	43	43
Individuals	421,042	421,042	421,042

Notes: This table adds additional controls to the recentered IV regressions in Table 2. The regressions in both panels add interactions of the 2012 state median income, year, and the Republican governor indicator. The regressions in Panel B additionally control for the interactions of 2012 state Medicaid coverage rates with those three variables. State-clustered standard errors are reported in parentheses; 95% confidence intervals, obtained by a wild score bootstrap, are reported in brackets.

Table A4: Medicaid Application: Recentered IV Including Non-Exposed Individuals

	Has Medicaid		Has Private Insurance		Has Employer-Sponsored Insurance	
	Recentered (1)	Controlled (2)	Recentered (3)	Controlled (4)	Recentered (5)	Controlled (6)
Panel A. Baseline Controls						
Eligible	0.032	0.072	0.193	0.061	0.208	0.071
	(0.085)	(0.038)	(0.290)	(0.123)	(0.301)	(0.127)
	[-0.433,0.148]	[-0.072,0.128]	[-0.214,1.796]	[-0.123,0.526]	[-0.204,1.907]	[-0.118,0.565]
Panel B. With Demographics × Post						
Eligible	0.116	0.116	−0.029	−0.030	−0.018	−0.019
	(0.012)	(0.011)	(0.013)	(0.012)	(0.012)	(0.012)
	[0.092,0.149]	[0.093,0.148]	[-0.052,0.003]	[-0.051,0.002]	[-0.040,0.012]	[-0.040,0.011]
Panel C. With Exposed Sample × Post						
Eligible	0.094	0.093	−0.012	−0.013	−0.005	−0.006
	(0.011)	(0.011)	(0.015)	(0.016)	(0.017)	(0.018)
	[0.064,0.118]	[0.059,0.118]	[-0.039,0.034]	[-0.041,0.038]	[-0.034,0.047]	[-0.037,0.052]
Exposed Sample	N	N	N	N	N	N
States	43	43	43	43	43	43
Individuals	2,397,313	2,397,313	2,397,313	2,397,313	2,397,313	2,397,313

Notes: Panel A of this table reports second-stage coefficients from versions of the recentered IV regressions in Table 2, estimated in the full sample of individuals in 2013–14. Columns 1, 3, and 5 use as an instrument a recentered prediction of Medicaid eligibility while Columns 2, 4, and 6 do not recenter but control for the expected prediction. All regressions control for state and year fixed effects and an indicator for Republican-governed states interacted with year. The regressions in Panel B additionally control for deciles of household income, interacted with indicators for parental and work status and year. The regressions in Panel C instead add the interaction of year and an indicator for an individual having characteristics that make them exposed to the partial ACA Medicaid expansion in 2014. State-clustered standard errors are reported in parentheses; 95% confidence intervals, obtained by a wild score bootstrap, are reported in brackets.